Chapter 4
Wave Propagation in a Stratified Medium: The Thin-Film Approach

4.1 Introduction

Chapter 4 changes gears, so to speak. It reviews a technique that uses the unitary state transition matrix for the system of first-order electromagnetic wave equations in a transmission medium that is thin, stratified, and linear [1–3]. This approach has proven useful for calculating the propagation of an electromagnetic wave through a thin film with Cartesian stratification. This chapter provides ready access to several propagation concepts that arise in the Mie scattering formulation: (1) the concepts of incoming and outgoing standing waves and their asymptotic forms, (2) turning points, (3) the osculating parameter technique in multiple Airy layers and the limiting forms of its solutions for a continuously varying refractivity, and (4) the accuracy of the osculating parameter technique within a Cartesian framework. The chapter also deals with the delicate problem of how to asymptotically match the incoming and outgoing solutions based on the osculating parameter technique. Sections 4.10 and 4.11 extend the thin-film approach and the unitary state transition matrix to cylindrical and spherical stratified media. Section 4.12 establishes a duality between spherical or cylindrical stratification and Cartesian stratification. This duality allows certain transformations to be applied to convert a problem with one type of stratification into another. The material in Chapters 5 and 6 does not depend on the material in this chapter. Therefore, this chapter may be skipped or skimmed. However, for the development of modified Mie scattering in Chapter 5, the background material herein may prove useful from time to time, particularly in the use of the Airy layer, a layer in which the gradient of the refractivity is a constant.

4.2 Thin-Film Concepts

Here we use the thin-film concepts [1–3] to develop the characteristic matrix, which describes the propagation of an electromagnetic wave through a stratified medium. When the "thin atmosphere" conditions hold [see Section 2.2, Eqs. (2.2-8) and (2.2-9)], this approach provides accurate results, and it also is instructive. In the usual thin-film approach, the stratified medium first is treated as a multi-layered medium. The index of refraction is held constant within each layer, but it is allowed to change across each boundary between the layers by an amount equal to some finite number (corresponding, for example, to the average gradient within the neighboring layers) times the thickness of the layer. Within each layer, the wave equations are readily solved in terms of sinusoid functions. The continuity conditions from Maxwell's equations allow one to tie the solutions together from neighboring layers across each layer boundary. Then the maximum thickness of each layer is driven to zero while the number of layers is allowed to grow indefinitely large so that the total thickness of the medium remains invariant. The resulting approximate solution from this ensemble of concatenated solutions can provide an accurate description of the electromagnetic field throughout the medium if the "thin atmosphere" conditions hold and if turning points are avoided.

4.2.1 Cartesian Stratification

We first develop the case for two-dimensional wave propagation in a medium with planar stratification. Many of the concepts developed here are applicable to the spherical stratified case, which we treat later. For the introduction of the Cartesian case, we follow closely the treatment presented in [3]. Here we assume that an electromagnetic wave, linearly polarized along the *y*-axis (see Fig. 4-1), i.e., a tranverse electric (TE) wave, is travelling through the medium. $\boldsymbol{H}$ lies in the *xz*-plane of incidence; $\boldsymbol{E}$ is parallel to the *y*-axis. $\boldsymbol{S}$ is the Poynting vector, also lying in the *xz*-plane. The angle φ is the angle of incidence of the wave. The wave is invariant in the y direction. The medium itself is stratified so that a planar surface of constant index of refraction is oriented perpendicular to the *x*-axis. The surfaces themselves are infinite in extent and parallel to the *yz*-plane. It will be sufficient to consider the TE case and its propagation in the *xz*-plane. Analyzing the TE case is preferable because in a medium where $\mu \equiv 1$ the equations are simpler [see Eq. (4.2-5)] than they are for the transverse magnetic (TM) case. To obtain results appropriate for the TM case, we can use the mathematical description for the $\boldsymbol{H}$ field that we will obtain for the TE case combined with use of the symmetry property in Maxwell's equations mentioned earlier. Maxwell's equations remain invariant when the definitions of the field vectors and their medium parameters are simultaneously exchanged according to the transformation

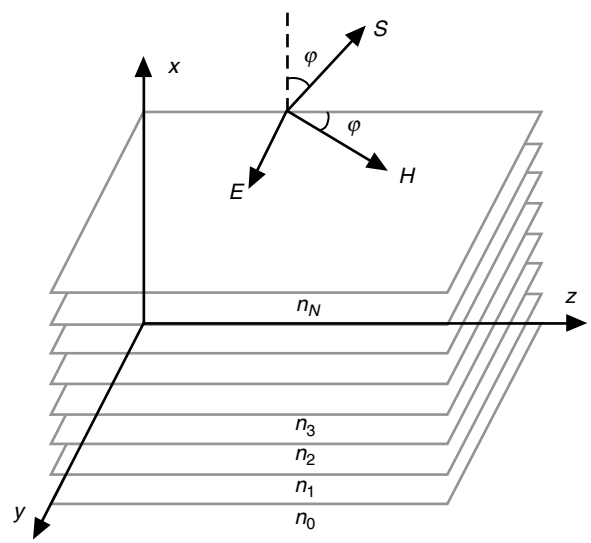

Fig. 4-1. Geometry for Cartesian stratification and a TE wave.

$(\boldsymbol{E},\boldsymbol{H},\varepsilon,\mu) \Leftrightarrow (-\boldsymbol{H},\boldsymbol{E},\mu,\varepsilon)$. This allows us to obtain $\boldsymbol{E}$ for the TM case from knowing $\boldsymbol{H}$ for the TE case.

In the plane of incidence of a planar wave, that is, the *xz*-plane, it follows for a TE wave that $E_x = E_z = 0$. The curl relations in Maxwell's equations for a TE harmonic wave give

$$\left.\begin{aligned} \frac{\partial E_y}{\partial x} &= ik\mu H_z \\ \frac{\partial E_y}{\partial z} &= -ik\mu H_x \\ H_y &= 0 \end{aligned}\right\} \tag{4.2-1a}$$

$$\left.\begin{aligned} \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= -ik\varepsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= 0 \\ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= 0 \end{aligned}\right\} \tag{4.2-1b}$$

It follows from these equations that $H_y \equiv 0$ and that H_x, H_z, and E_y are functions of *x* and *z* only. From Eq. (3.2-1), it follows that the time-independent

component of the electric field for the TE case must satisfy the modified Helmholtz equation

$$\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial z^2}-\frac{d(\log\mu)}{dx}\frac{\partial E_y}{\partial x}+n^2k^2E_y=0 \tag{4.2-2}$$

Using the separation of variables technique, we set as a trial solution

$$E_y(z,x)=U(x)Z(z) \tag{4.2-3}$$

Then Eq. (4.2-2) becomes

$$\frac{1}{U}\frac{d^2U}{dx^2}+n^2k^2-\frac{d(\log\mu)}{dx}\frac{1}{U}\frac{dU}{dx}=\frac{1}{Z}\frac{d^2Z}{dz^2} \tag{4.2-4}$$

The left-hand side (LHS) of Eq. (4.2-4) is a function only of x while the right-hand side (RHS) is a function only of z. This can be true only if both sides are constant. Hence,

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=-k^2n_o^2;\quad n_o=\text{constant} \tag{4.2-5}$$

Here n_o is a constant to be determined later. The solution for $Z(z)$ follows immediately:

$$Z=Z_0\exp(\pm ikn_oz) \tag{4.2-6}$$

Thus, kn_o is the rate of phase accumulation of the time-independent component of the wave along the z-direction, and it is an invariant for a particular wave. Without loss of generality, we can assume that the wave is trending from left to right in Fig. 4-1 (i.e., in the direction of positive z); hence, we adopt the positive sign in Eq. (4.2.6). Then the electric field for a harmonic wave is given by

$$E_y=U(x)\exp\big(i(kn_oz-\omega t)\big) \tag{4.2-7}$$

From Maxwell's equations [see Eq. (4.2-1a), it follows that the magnetic field components are expressed in terms of different functions of x but the same functions of z and t. These are given by

$$\left.\begin{aligned}H_z&=V(x)\exp\big(i(kn_oz-\omega t)\big)\\H_x&=W(x)\exp\big(i(kn_oz-\omega t)\big)\end{aligned}\right\} \tag{4.2-8}$$

The wave equations also require that the U, V, and W functions satisfy certain conditions among themselves, which from Eq. (4.2-1) are given by

$$\left.\begin{aligned} U' &= ik\mu V \\ V' &= ik\left(n_o W + \varepsilon U\right) \\ \mu W + n_o U &= 0 \end{aligned}\right\} \tag{4.2-9}$$

Eliminating W, we obtain a coupled system of first-order differential equations for U and V:

$$\left.\begin{aligned} U' &= ik\mu V \\ V' &= ik\mu^{-1}\left(n^2 - n_o^2\right)U \end{aligned}\right\} \tag{4.2-10}$$

Alternatively, one can convert this coupled system into a pair of independent second-order differential equations, which from Eq. (3.2-1) are given by

$$\left.\begin{aligned} \frac{d^2U}{dx^2} - \frac{d(\log\mu)}{dx}\frac{dU}{dx} + k^2\left(n^2 - n_o^2\right)U &= 0 \\ \frac{d^2V}{dx^2} - \frac{d\left(\log\left[\left(n^2 - n_o^2\right)/\mu\right]\right)}{dx}\frac{dV}{dx} + k^2\left(n^2 - n_o^2\right)V &= 0 \end{aligned}\right\} \tag{4.2-11}$$

For the TM case ($H_x = H_z \equiv 0$), the transformation $(\boldsymbol{E}, \boldsymbol{H}, \varepsilon, \mu) \Leftrightarrow (-\boldsymbol{H}, \boldsymbol{E}, \mu, \varepsilon)$ yields

$$H_y = U(x)\exp\left(i\left(kn_o z - \omega t\right)\right) \tag{4.2-7′}$$

and

$$\left.\begin{aligned} E_z &= -V(x)\exp\left(i\left(kn_o z - \omega t\right)\right) \\ E_x &= -W(x)\exp\left(i\left(kn_o z - \omega t\right)\right) \end{aligned}\right\} \tag{4.2-8′}$$

The wave equations for the TM case become

$$\left.\begin{aligned} U' &= ik\varepsilon V \\ V' &= ik\varepsilon^{-1}\left(n^2 - n_o^2\right)U \\ \varepsilon W + n_o U &= 0 \end{aligned}\right\} \tag{4.2-9′}$$

or

$$\left.\begin{aligned}\frac{d^2U}{dx^2}-\frac{d\log\varepsilon}{dx}\frac{dU}{dx}+k^2\left(n^2-n_o^2\right)U=0\\ \frac{d^2V}{dx^2}-\frac{d\left(\log\left[\left(n^2-n_o^2\right)/\varepsilon\right]\right)}{dx}\frac{dV}{dx}+k^2\left(n^2-n_o^2\right)V=0\end{aligned}\right\}\qquad(4.2\text{-}11')$$

Returning to the TE case, we note that in general U, V, and W are complex. From Eq. (4.2-7), a surface of constant phase for E_y (called the cophasal surface) is defined by

$$\psi(x)+kn_oz-\omega t=\text{constant}\qquad(4.2\text{-}12)$$

where $\psi(x)$ is the phase of $U(x)$. For an infinitesimal displacement $(\delta x,\delta z)$ at a fixed time and lying on the cophasal surface, we have from Eq. (4.2-12) the condition $\psi'\delta x+kn_o\delta z=0$. Therefore, the angle of incidence φ that the cophasal surface makes with the yz-plane (Fig. 4-1) is given by

$$\tan\varphi=-\delta x/\delta z=kn_o/\psi'\qquad(4.2\text{-}13)$$

For the special case where the wave is planar, we have $\psi'(x)=kn\cos\varphi$, from which it follows that the constant, n_o, in Eq. (4.2-5) is given by

$$n_o=n\sin\varphi=\text{constant}\qquad(4.2\text{-}14)$$

which is Snell's law. It follows that the condition $n_o=\text{constant}$, obtained from the solution to the modified wave equation in Eq. (4.2-5), can be considered as a generalization of Snell's law. The value n_o, not to be confused with n_0 associated with the index of refraction of the 0th layer in Fig. 4-1, provides the index of refraction and therefore the layer(s) in which, according to geometric optics, $\varphi=\pi/2$, which marks a turning point for the wave.

4.3 The Characteristic Matrix

Returning to the coupled system in Eq. (4.2-10), we know that U and V both have two independent solutions to Eq. (4.2-11). Let these solutions be given by $F(x,x_0)$ and $f(x,x_0)$ for U, and by $G(x,x_0)$ and $g(x,x_0)$ for V. However, these solutions are constrained by the conditions in Eq. (4.2-10), which are given by

$$\left.\begin{aligned} &\frac{dF}{dx} = ik\mu G, \quad \frac{df}{dx} = ik\mu g \\ &\frac{dG}{dx} = ik\left(\varepsilon - n_o^2\mu^{-1}\right)F, \ \frac{dg}{dx} = ik\left(\varepsilon - n_o^2\mu^{-1}\right)f \end{aligned}\right\} \tag{4.3-1}$$

Using a Green function-like approach, we construct these solutions so that the following specific boundary values are obtained:

$$\left.\begin{aligned} &F(x_0, x_0) = 1, \quad f(x_0, x_0) = 0 \\ &G(x_0, x_0) = 0, \ g(x_0, x_0) = 1 \end{aligned}\right\} \tag{4.3-2}$$

Then it follows that in matrix form U and V can be written as

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = \begin{bmatrix} F(x, x_0) & f(x, x_0) \\ G(x, x_0) & g(x, x_0) \end{bmatrix} \begin{pmatrix} U(x_0) \\ V(x_0) \end{pmatrix} \tag{4.3-3}$$

We define the characteristic matrix $\boldsymbol{M}[x, x_0]$ by

$$\boldsymbol{M}[x, x_0] = \begin{bmatrix} F(x, x_0) & f(x, x_0) \\ G(x, x_0) & g(x, x_0) \end{bmatrix} \tag{4.3-4}$$

Hence, Eq. (4.3-3) shows that the description of the electromagnetic wave through the stratified medium is borne solely by the initial conditions and by this state transition matrix $\boldsymbol{M}[x, x_0]$, a 2×2 unitary matrix. From the theory of ordinary differential equations, one can show that $\boldsymbol{M}[x, x_0]$ has a constant determinant, and in the special case where the boundary values given in Eq. (4.3-2) apply, $\mathrm{Det}\left[\boldsymbol{M}[x, x_0]\right] = 1$ for all values of x. This can be shown to result from the conservation of energy principle that applies to a non-absorbing medium where n is real. Henceforth, we will focus our attention on the properties of $\boldsymbol{M}[x, x_0]$ for different cases of stratification in the propagation medium.

4.4 The Stratified Medium as a Stack of Discrete Layers

An important transitive property of $\boldsymbol{M}[x, x_0]$ is obtained from the following observation. Consider two contiguous layers of different indices of refraction. The thickness of the first layer is $x_1 - x_0$, and its index of refraction is given by $n_1(x)$. The thickness of the second layer is $x_2 - x_1$, and its index of refraction is given by $n_2(x)$. Across a surface, Maxwell's equations require the tangential components of $\boldsymbol{E}$ to be continuous, and they also require the

tangential components of $\boldsymbol{H}$ to be continuous when surface currents are absent, which is assumed here. Since U and V describe the tangential components of the electromagnetic field vectors, U and V also must be continuous across the boundary. Hence, the relationships for the electromagnetic field are given by

$$\left.\begin{aligned}
&\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \boldsymbol{M}[x_2, x_1]\begin{pmatrix} U_1 \\ V_1 \end{pmatrix}, \ \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \boldsymbol{M}[x_1, x_0]\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \\
&\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \boldsymbol{M}[x_2, x_1]\boldsymbol{M}[x_1, x_0]\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \boldsymbol{M}[x_2, x_0]\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \\
&\therefore \ \boldsymbol{M}[x_2, x_0] = \boldsymbol{M}[x_2, x_1]\boldsymbol{M}[x_1, x_0]
\end{aligned}\right\} \tag{4.4-1}$$

This product rule can be generalized to N layers by

$$\boldsymbol{M}[x_N, x_0] = \boldsymbol{M}[x_N, x_{N-1}]\boldsymbol{M}[x_{N-1}, x_{N-2}]\cdots\boldsymbol{M}[x_1, x_0] = \prod_{k=1}^{N}\boldsymbol{M}[x_k, x_{k-1}] \tag{4.4-2}$$

If the form of the index of refraction $n(x)$ within each layer is such that the solutions for U and V can be expressed in terms of relatively simple functions, then in some cases it is possible to obtain a closed form for $\boldsymbol{M}[x, x_0]$ using the product rule. Approximate forms of sufficient accuracy also can be obtained in some cases.

4.4.1 The Characteristic Matrix when $n(x)$ = constant

When the index of refraction is constant within a layer, we obtain sinusoid solutions to Eqs. (4.2-10) and (4.2-11) for the TE case, which can be forced to satisfy the boundary conditions in Eq. (4.3-2). These solutions are the elements of the characteristic matrix that describes a plane wave traversing the layer. The functional elements of the characteristic matrix are given by

$$\left.\begin{aligned}
&F(x, x_0) = \cos[k\varpi(x - x_0)], \quad f(x, x_0) = i\frac{1}{\varpi}\sin[k\varpi(x - x_0)] \\
&G(x, x_0) = i\varpi\sin[k\varpi(x - x_0)], \quad g(x, x_0) = \cos[k\varpi(x - x_0)] \\
&\varpi = \mu^{-1}\sqrt{n^2 - n_o^2} = \sqrt{\frac{\varepsilon}{\mu}}\cos\varphi
\end{aligned}\right\} \tag{4.4-3}$$

Here φ is the angle of incidence of the plane wave in the layer (Fig. 4-1). It follows that $k\mu\varpi$ is the rate of phase accumulation of the time-independent component of the wave along the x-axis, perpendicular to the plane of stratification.

For the TM case, the solutions are of the same form except $f_{\mathrm{TM}} = (\varepsilon / \mu) f_{\mathrm{TE}}$ and $G_{\mathrm{TM}} = (\mu / \varepsilon) G_{\mathrm{TE}}$.

4.4.2 A Stack of Homogeneous Layers when $n(x)$ is Piecewise Constant

We apply these results to a stack of layers as shown in Fig. 4-1. The index of refraction varies from layer to layer, but within the jth layer the index of refraction is constant so that $\varpi_j = \mu_j^{-1}\left(n_j^2 - n_o^2\right)^{1/2}$ also is a constant within this layer. We also note that $\varpi_j = \left(\varepsilon_j / \mu_j\right)^{1/2} \cos\varphi_j$, where φ_j is the angle of incidence of the wave within the jth layer. Thus, within the jth layer the angle of incidence and the rate of phase accumulation remain constant.

Let us now define a reference characteristic matrix $\tilde{\boldsymbol{M}}$ by

$$\tilde{\boldsymbol{M}}\left[x_j, x_{j-1}\right] = \begin{bmatrix} \cos\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] & \dfrac{i}{\overline{\varpi}}\sin\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] \\ i\overline{\varpi}\sin\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] & \cos\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] \end{bmatrix} \tag{4.4-4}$$

where $\overline{\varpi}$ is a constant across all layers; its value will be set later. Here, x_j marks the upper boundary (Fig. 4-1) of the jth layer. We set

$$\boldsymbol{M}\left[x_j, x_{j-1}\right] = \tilde{\boldsymbol{M}}\left[x_j, x_{j-1}\right] + \delta\boldsymbol{M}\left[x_j, x_{j-1}\right] \tag{4.4-5}$$

where $\delta\boldsymbol{M}$ is defined as the difference between the actual characteristic matrix $\boldsymbol{M}\left[x_j, x_{j-1}\right]$ and the reference matrix $\tilde{\boldsymbol{M}}\left[x_j, x_{j-1}\right]$. This difference is due to $\delta\varpi_j = \varpi_j - \overline{\varpi}$. Then to first order in $\delta\varpi_j$, $\delta\boldsymbol{M}$ is given by

$$\delta\boldsymbol{M}\left[x_j, x_{j-1}\right] \doteq \begin{bmatrix} 0 & -\dfrac{i}{\overline{\varpi}}\sin\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] \\ i\overline{\varpi}\sin\left[k\varpi_j\left(x_j - x_{j-1}\right)\right] & 0 \end{bmatrix} \frac{\delta\varpi_j}{\overline{\varpi}} \tag{4.4-6}$$

Here $\delta\varpi_j$ is assumed to be small but not negligible.

From the product rule in Eq. (4.4-2) and truncating to first order, it follows that

$$\begin{aligned}\boldsymbol{M}[x_N, x_0] = \boldsymbol{M}_{N,0} &= \prod_{j=1}^{N}\left(\tilde{\boldsymbol{M}}_{j,j-1} + \delta\boldsymbol{M}_{j,j-1}\right) \\ &\doteq \prod_{j=1}^{N}\tilde{\boldsymbol{M}}_{j,j-1} + \sum_{j=2}^{N-1}\left(\left(\prod_{l=j}^{N-1}\tilde{\boldsymbol{M}}_{l+1,l}\right)\delta\boldsymbol{M}_{j,j-1}\left(\prod_{l=1}^{j-1}\tilde{\boldsymbol{M}}_{l,l-1}\right)\right) \\ &\quad + \delta\boldsymbol{M}_{N,N-1}\left(\prod_{l=1}^{N-1}\tilde{\boldsymbol{M}}_{l,l-1}\right) + \left(\prod_{l=2}^{N}\tilde{\boldsymbol{M}}_{l,l-1}\right)\delta\boldsymbol{M}_{1,0}\end{aligned} \tag{4.4-7}$$

Truncating the expansions in Eqs. (4.4-6) and (4.4-7) to first order in $\delta\varpi_j$ and $\delta\boldsymbol{M}$ should be accurate if $|\varpi'|$ is sufficiently small throughout the stack. The range of validity is discussed later. With regard to the reference characteristic matrix, it is easily shown that

$$\tilde{\boldsymbol{M}}_{m,l} = \prod_{j=l+1}^{m}\tilde{\boldsymbol{M}}_{j,j-1} = \begin{bmatrix}\cos\mathscr{A}_{m,l} & \dfrac{i}{\overline{\varpi}}\sin\mathscr{A}_{m,l} \\ i\overline{\varpi}\sin\mathscr{A}_{m,l} & \cos\mathscr{A}_{m,l}\end{bmatrix} \tag{4.4-8}$$

where

$$\left.\begin{aligned}\mathscr{A}_{m,l} &= k\sum_{j=l+1}^{m}\varpi_j\left(x_j - x_{j-1}\right) \\ \mathscr{A}_{m,m} &= 0\end{aligned}\right\} \tag{4.4-9}$$

Also, defining $\tilde{\boldsymbol{M}}_{N,N} = \tilde{\boldsymbol{M}}_{0,0} = \boldsymbol{I}$, the identity matrix, the jth product in Eq. (4.4-7) for $j = 1, 2, \cdots, N$, becomes

$$\begin{aligned}\left(\prod_{l=j+1}^{N}\tilde{\boldsymbol{M}}_{l,l-1}\right)\delta\boldsymbol{M}_{j,j-1}\left(\prod_{l=1}^{j-1}\tilde{\boldsymbol{M}}_{l,l-1}\right) &= \tilde{\boldsymbol{M}}_{N,j}\delta\boldsymbol{M}_{j,j-1}\tilde{\boldsymbol{M}}_{j-1,0} \\ &= \frac{\delta\varpi_j}{\overline{\varpi}}\sin\left[k\varpi_j\left(x_j - x_{j-1}\right)\right]\begin{bmatrix}-\sin\mathscr{B}_j & \dfrac{i}{\overline{\varpi}}\cos\mathscr{B}_j \\ -i\overline{\varpi}\cos\mathscr{B}_j & \sin\mathscr{B}_j\end{bmatrix}\end{aligned} \tag{4.4-10}$$

where

$$\mathscr{B}_j = \sum_{l=j+1}^{N}k\varpi_l\left(x_l - x_{l-1}\right) - \sum_{l=1}^{j-1}k\varpi_l\left(x_l - x_{l-1}\right) \tag{4.4-11}$$

Now we go to the limit, allowing $x_j - x_{j-1} \to 0$ and $N \to \infty$ so that

$$\sum_{j=1}^{N} k\varpi_j\left(x_j - x_{j-1}\right) \to k\int_{x_0}^{x_F} \varpi dx \tag{4.4-12}$$

It follows that

$$\left.\begin{aligned}
&\mathscr{A}_{N,0} \to \mathscr{A}\left(x_F, x_0\right) = \int_{x_0}^{x_F} \varpi dx \\
&\mathscr{B}_j \to \mathscr{B}(x) = k\int_{x}^{x_F} \varpi dx - k\int_{x_0}^{x} \varpi dx = \mathscr{A}\left(x_F, x\right) - \mathscr{A}\left(x, x_0\right) \\
&\varpi(x) = \mu^{-1}\sqrt{n^2(x) - n_o^2}, \quad n_o = n(x)\big|_{x=x_o}
\end{aligned}\right\} \tag{4.4-13}$$

where x_F is the final value of x, nominally where the electromagnetic field is to be evaluated. The quantity $\mathscr{A}\left(x_F, x_0\right)$ is the total phase accumulation of the time-independent component of the wave along the x-direction between x_0 and x_F. Note that $\mathscr{A}\left(x_F, x_0\right)$ is an implicit function of the refractivity profile of the medium, and it also is a function of n_o, (through ϖ) or the angle of incidence, for a specific wave.

Upon passing to the limit and integrating, Eq. (4.4-10) becomes

$$\begin{aligned}
&\sum_{j=1}^{N} \tilde{\boldsymbol{M}}_{N,j}\delta\boldsymbol{M}_{j,j-1}\tilde{\boldsymbol{M}}_{j-1,0} \to \int_{x_0}^{x_F} \tilde{\boldsymbol{M}}\left[x_F, x\right]\delta\boldsymbol{M}\left[x, x\right]\tilde{\boldsymbol{M}}\left[x, x_0\right]dx = \\
&\frac{k}{\overline{\varpi}}\int_{x_0}^{x_F} \left(\varpi(x) - \overline{\varpi}\right)\varpi(x)\begin{bmatrix} -\sin\mathscr{B}(x) & -\dfrac{i}{\overline{\varpi}}\cos\mathscr{B}(x) \\ i\overline{\varpi}\cos\mathscr{B}(x) & \sin\mathscr{B}(x) \end{bmatrix}dx
\end{aligned} \tag{4.4-14}$$

Upon noting that $\mathscr{B}'(x) = -2k\varpi(x)$, we can integrate this integral by parts to obtain

$$\begin{aligned}
&\int_{x_0}^{x_F} \tilde{\boldsymbol{M}}\left[x_F, x\right]\delta\boldsymbol{M}\left[x, x\right]\tilde{\boldsymbol{M}}\left[x, x_0\right]dx = \\
&\frac{1}{2\overline{\varpi}}\begin{bmatrix} \left(\varpi_0 - \varpi_F\right)\cos\mathscr{A} & \dfrac{i}{\overline{\varpi}}\left(2\overline{\varpi} - \varpi_0 - \varpi_F\right)\sin\mathscr{A} \\ i\overline{\varpi}\left(\varpi_0 + \varpi_F - 2\overline{\varpi}\right)\sin\mathscr{A} & \left(\varpi_F - \varpi_0\right)\cos\mathscr{A} \end{bmatrix} + \begin{bmatrix} I_1 & -\dfrac{i}{\overline{\varpi}}I_2 \\ i\overline{\varpi}I_2 & -I_1 \end{bmatrix}
\end{aligned} \tag{4.4-14′}$$

where

$$\left.\begin{aligned} I_1 &= \frac{1}{2\overline{\varpi}}\int_{x_0}^{x_F}\frac{d\varpi}{dx}\cos\mathscr{B}(\zeta)dx \\ I_2 &= \frac{1}{2\overline{\varpi}}\int_{x_0}^{x_F}\frac{d\varpi}{dx}\sin\mathscr{B}(\zeta)dx \end{aligned}\right\} \tag{4.4-15}$$

We now set $\overline{\varpi}$ equal to its "average" value over the interval $[x_F, x_0]$. That is,

$$\overline{\varpi} = (\varpi_F + \varpi_0)/2 \tag{4.4-16}$$

Adding the resulting perturbation matrix in Eq. (4.4-14′) to the reference matrix $\tilde{\boldsymbol{M}}[x_F, x_0]$ given by Eq. (4.4-8), we obtain a first-order expression for the characteristic matrix $\boldsymbol{M}[x_F, x_0]$ applicable to the entire stratified medium for the TE case. This is given by

$$\boldsymbol{M}[x_F, x_0] \doteq \begin{bmatrix} \dfrac{\varpi_0}{\overline{\varpi}}\cos\mathscr{A} + I_1 & \dfrac{i}{\overline{\varpi}}(\sin\mathscr{A} - I_2) \\ \\ i\overline{\varpi}(\sin\mathscr{A} + I_2) & \dfrac{\varpi_F}{\overline{\varpi}}\cos\mathscr{A} - I_1 \end{bmatrix} \tag{4.4-17}$$

4.4.3 Range of Validity

Let us estimate the range of validity of the linear perturbation approach used in Eq. (4.4-7) to obtain Eq. (4.4-17). We have noted that its accuracy will depend on $|\varpi'|$ being sufficiently small. From Eqs. (4.2-14) and (4.4-3), it follows that

$$\varpi' = n'\sec\varphi \tag{4.4-18}$$

The magnitude of the first term on the RHS of Eq. (4.4-14′) is of the order of $(x_F - x_0)\langle n'\sec\varphi\rangle$, where $\langle\ \rangle$ denotes an average over the interval $(x_F - x_0)$. Thus, if this term is small, the linear truncation should be valid. For the second term on the RHS of Eq. (4.4-14') involving the I integrals in Eq. (4.4-15), let us assume that ϖ' is a constant over the integration interval. In this case, $\mathscr{B}(x)$ is quadratic in x and Eq. (4.4-15) involves Fresnel integrals. It is easily shown that $|I_1| \approx |I_2| \approx (\varpi'\lambda)^{1/2}/\varpi$ for $(x_F - x_0)/\lambda >> 1$; these terms away from turning points are generally small for "thin atmospheres," and smaller than the first term. Although the I integrals will be small under these conditions, their integrands are highly oscillatory when $(x_F - x_0)/\lambda >> 1$. If retention of these terms is necessary, special integration algorithms using the rapid variation of $\exp[i\mathscr{B}(x)]$ and the slowly varying character of $d\varpi/dx$ are helpful.

We conclude for n' sufficiently small and for points located sufficiently far from turning points, where $\varphi = \pi/2$, that the linear truncation used in Eq. (4.4-7) will be sufficiently accurate. If these conditions hold, one can neglect the $N^2/2$ second-order terms $\delta\boldsymbol{M}_{j,j-1}\delta\boldsymbol{M}_{m,m-1}$ in the product rule expansion in Eq. (4.4-7), as well as the second-order terms in Eq. (4.4-6).

We note a special interpretation for the quantity $n'(x_F - x_0) = (n'x_0)((x_F - x_0)/x_0)$. In spherical coordinates, the first product is the ratio of the radius of curvature of the refracting surface to the radius of curvature of the ray path, which is one measure of atmospheric "thinness." For dry air at the Earth's surface, this quantity is about 1/4. The second product is merely the fraction of the total radius traversed by the ray, usually very small for a large sphere.

4.4.4 The TM Case

It is easy to show using the transformation $(\boldsymbol{E},\boldsymbol{H},\varepsilon,\mu) \Leftrightarrow (-\boldsymbol{H},\boldsymbol{E},\mu,\varepsilon)$ that the TM version of Eq. (4.4-17) is given by

$$\left.\begin{aligned} \boldsymbol{M}[x_F,x_0] &\doteq \begin{bmatrix} \dfrac{\varpi_{\mathrm{TM}}\big|_{x_F}}{\overline{\varpi}_{\mathrm{TM}}}\cos\mathscr{A}_{\mathrm{TM}} + I_1 & \dfrac{i}{\overline{\varpi}_{\mathrm{TM}}}\left(\sin\mathscr{A}_{\mathrm{TM}} + I_2\right) \\ i\overline{\varpi}_{\mathrm{TM}}\left(\sin\mathscr{A}_{\mathrm{TM}} - I_2\right) & \dfrac{\varpi_{\mathrm{TM}}\big|_{x_0}}{\overline{\varpi}_{\mathrm{TM}}}\cos\mathscr{A}_{\mathrm{TM}} - I_1 \end{bmatrix} \\ \varpi_{\mathrm{TM}} &= \frac{\mu}{\varepsilon}\varpi_{\mathrm{TE}} = \sqrt{\frac{\mu}{\varepsilon}}\cos\varphi \end{aligned}\right\} \quad (4.4\text{-}17')$$

The form for $\boldsymbol{M}[x_F,x_0]$ in Eq. (4.4-17), but without the I_1 and I_2 terms, first appears in [4]. It also can be obtained by applying the Wentzel–Kramer–Brillouin (WKB) method to Eq. (4.2-11). The WKB solution to Eq. (4.2-11) was almost certainly known during Lord Rayleigh's time because of his studies of acoustic waves in a refracting medium.

One could generalize this problem to include a stratified medium with an embedded discontinuity. Within the medium a surface parallel to the stratification is embedded. This surface acts as a boundary between two regions. Within each region, $n(x)$ is continuous, but across the boundary $n(x)$ or its gradient is discontinuous. Within each region, a characteristic matrix of the form in Eq. (4.4-17) applies, and the product of these two matrices provides the characteristic matrix that spans the entire medium, including the discontinuity. One also can calculate the reflection and transmission

coefficients across the discontinuous boundary in terms of the elements of the characteristic matrices at the boundary. Since a modified Mie scattering approach will be used in Chapter 5 to address the problem of a scattering spherical surface embedded in a refracting medium, it will not be pursued further here. See [3] and [5] for further discussion of this case.

4.5 The Characteristic Matrix for an Airy Layer

We can check the characteristic matrix given in Eq. (4.4-17), which results from a linear theory applied to an infinite stack of infinitesimal layers, with an essentially exact result, which can be obtained when the gradient of the index of refraction is constant within the medium. We designate a layer with a constant gradient an "Airy layer," because the Airy functions form the solution set for such a layer. We let

$$n^2 = n_0^2 + 2n_0 n'(x - x_0) \tag{4.5-1}$$

where n_0 and n' are constants throughout the layer and $n'(x_F - x_0)$ is sufficiently small so the term $\left(n'(x_F - x_0)\right)^2$ can be neglected. This quasi-linear form for the index of refraction has application in atmospheric propagation studies. There the continuous profile for $n(x)$ is approximated by a series of piecewise constant-gradient segments [6].

Returning to Eqs. (4.2-10) and (4.2-11), we have for the TE case in a single layer:

$$\left.\begin{aligned} &\frac{d^2U}{dx^2} + k^2\left(n_0^2 - n_o^2 + 2n_0 n'(x - x_0)\right)U = 0 \\ &\frac{dU}{dx} = ik\mu V \end{aligned}\right\} \tag{4.5-2}$$

Without loss of generality through reorientation of our coordinate frame, we can assume that $n' \geq 0$. Next, we make the transformation

$$\hat{y} = -\gamma^{-2}\left(n^2 - n_o^2\right) = -\varpi_0^2\gamma^{-2} - k\gamma(x - x_0) \tag{4.5-3}$$

where the constants ϖ_0 and γ are given by

$$\left.\begin{aligned} &\varpi_0^2 = n_0^2 - n_o^2 \\ &\gamma = \left(2k^{-1}n_0 n'\right)^{1/3} \end{aligned}\right\} \tag{4.5-4}$$

Equation (4.5-4) allows the possibility of ϖ_0^2 being negative; thus, ϖ_0 would be imaginary in this case. We show later that a negative value for ϖ_0^2 corresponds to a region where quantum tunneling applies; there $\hat{y}$ is positive and the amplitude of the electromagnetic field exponentially decays with increasing $\hat{y}$.

With this transformation in Eq. (4.5-3), the wave equations in Eq. (4.5-2) become

$$\left.\begin{aligned} &\frac{d^2U}{d\hat{y}^2} - \hat{y}U = 0 \\ &\frac{dU}{d\hat{y}} + i\gamma^{-1}V = 0 \end{aligned}\right\} \tag{4.5-2′}$$

Here we have set $\mu \equiv 1$. The solutions to these differential equations are the Airy functions and their derivatives; that is,

$$\left.\begin{aligned} &U(\hat{y}) = \left\{\mathrm{Ai}[\hat{y}], \mathrm{Bi}[\hat{y}]\right\} \\ &V(\hat{y}) = i\gamma\left\{\mathrm{Ai}'[\hat{y}], \mathrm{Bi}'[\hat{y}]\right\} \end{aligned}\right\} \tag{4.5-5}$$

In matrix form, the solutions are given by

$$\begin{pmatrix} U \\ V \end{pmatrix} = \boldsymbol{M}[\hat{y}, \hat{y}_0]\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \tag{4.5-6a}$$

The elements of the characteristic matrix $\boldsymbol{M}[\hat{y}, \hat{y}_0]$ are given by

$$\begin{aligned} \boldsymbol{M}[\hat{y}, \hat{y}_0] &= \begin{bmatrix} F(\hat{y}, \hat{y}_0) & f(\hat{y}, \hat{y}_0) \\ G(\hat{y}, \hat{y}_0) & g(\hat{y}, \hat{y}_0) \end{bmatrix} \\ &= \pi\begin{bmatrix} \begin{vmatrix} \mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}] \\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0] \end{vmatrix} & \dfrac{i}{\gamma}\begin{vmatrix} \mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}] \\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0] \end{vmatrix} \\ i\gamma\begin{vmatrix} \mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}] \\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0] \end{vmatrix} & -\begin{vmatrix} \mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}] \\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0] \end{vmatrix} \end{bmatrix} \end{aligned} \tag{4.5-6b}$$

where $\hat{y}$ is given in terms of x through Eq. (4.5-3) and $\hat{y}(x_0) = \hat{y}_0$. We note that $Fdg/d\hat{y} - fdG/d\hat{y} = 0$, $Gdf/d\hat{y} - gdF/d\hat{y} = 0$, or $d(Fg - Gf)/d\hat{y} = 0$, which is a consequence of the determinant of $\boldsymbol{M}[\hat{y}, \hat{y}_0]$ being a constant. We have

used the Wronskian of $\mathrm{Ai}[\hat{y}]$ and $\mathrm{Bi}[\hat{y}]$, $\mathrm{Ai}[\hat{y}]\mathrm{Bi}'[\hat{y}]-\mathrm{Ai}'[\hat{y}]\mathrm{Bi}[\hat{y}]=\pi^{-1}$, so that $\boldsymbol{M}\left[\hat{y}_0,\hat{y}_0\right]=\boldsymbol{I}$.

For this special case where the index of refraction is given by Eq. (4.5-1), we have

$$\varpi^2(x)=n^2(x)-n_o^2=\varpi_0^2+k\gamma^3\left(x-x_0\right)=-\gamma^2\hat{y} \tag{4.5-7}$$

and also

$$\frac{d\varpi}{dx}=k\gamma^2\left(-4\hat{y}\right)^{-1/2} \tag{4.5-8}$$

and

$$k\int_{x_0}^{x}\varpi(\zeta)d\zeta=\frac{2}{3}\left(\left(-\hat{y}\right)^{3/2}-\left(-\hat{y}_0\right)^{3/2}\right) \tag{4.5-9}$$

When $\hat{y}<<0$, $\hat{y}$ may be interpreted in terms of the angle of incidence φ by $\pm\gamma\left(-\hat{y}\right)^{1/2}=n\cos\varphi$. In geometric optics, $\hat{y}=\hat{y}_o\equiv 0$ corresponds to a turning point where $\varphi=\pi/2$. We will discuss turning points in more detail later for the TE wave.

When $\left(\varpi_0/\gamma\right)^2>>-k\gamma\left(x-x_0\right)$, then $y\le y_0<<0$ and we can use the negative argument asymptotic forms for the Airy functions given in Eq. (3.8-4) [7]. The elements of the characteristic matrix become

$$\left.\begin{aligned}&F\left(\hat{y},\hat{y}_0\right)\to\left(\frac{\hat{y}_0}{\hat{y}}\right)^{1/4}\cos\left(X-X_0\right),\ f\left(\hat{y},\hat{y}_0\right)\to\frac{i}{\gamma}\left(\frac{1}{\hat{y}\hat{y}_0}\right)^{1/4}\sin\left(X-X_0\right)\\&G\left(\hat{y},\hat{y}_0\right)\to i\gamma\left(\hat{y}\hat{y}_0\right)^{1/4}\sin\left(X-X_0\right),\ g\left(\hat{y},\hat{y}_0\right)\to\left(\frac{\hat{y}}{\hat{y}_0}\right)^{1/4}\cos\left(X-X_0\right)\end{aligned}\right\} \tag{4.5-10}$$

where from Eq. (4.5-9) we have

$$\left.\begin{aligned}&X=2/3\left(-\hat{y}\right)^{3/2}+\pi/4,\ X-X_0=\mathscr{H}\left(x,x_0\right)=k\int_{x_0}^{x}\varpi(u)du,\\&\hat{y}=-\gamma^2\varpi^2(x),\ \hat{y}\left(x_0\right)=\hat{y}_0=-\gamma^{-2}\left(n_0^2-n_o^2\right),\ \hat{y}\left(x_o\right)=\hat{y}_o=0,\\&\varpi\left(x_0\right)=\varpi_0\end{aligned}\right\} \tag{4.5-11}$$

Thus, the asymptotic form for the characteristic matrix for negative values of $\hat{y}$ becomes

$$\boldsymbol{M}[\hat{y},\hat{y}_0] \sim \begin{bmatrix} \left(\dfrac{\varpi_0}{\varpi_F}\right)^{1/2} \cos\mathscr{A} & \dfrac{i}{(\varpi_0\varpi_F)^{1/2}} \sin\mathscr{A} \\ i(\varpi_0\varpi_F)^{1/2} \sin\mathscr{A} & \left(\dfrac{\varpi_F}{\varpi_0}\right)^{1/2} \cos\mathscr{A} \end{bmatrix} \tag{4.5-12}$$

If we define the mean value of ϖ as $\overline{\varpi} = (\varpi_F\varpi_0)^{1/2}$, then Eq. (4.5-12) becomes

$$\boldsymbol{M}[\hat{y},\hat{y}_0] \sim \begin{bmatrix} \dfrac{\varpi_0}{\overline{\varpi}} \cos\mathscr{A} & \dfrac{i}{\overline{\varpi}} \sin\mathscr{A} \\ i\overline{\varpi} \sin\mathscr{A} & \dfrac{\varpi_F}{\overline{\varpi}} \cos\mathscr{A} \end{bmatrix} \tag{4.5-13}$$

It follows (away from turning points) that $\boldsymbol{M}[\hat{y}_F,\hat{y}_0]$ in Eq. (4.5-13) matches to first order in $(\varpi_F - \varpi_0)$ the characteristic matrix given in Eq. (4.4-17), which is derived from a more general but linear theory based on the thin-film approach.

Near a turning point, the limiting form in Eq. (4.5-13) breaks down when $\hat{y} >\sim -2$, or at $x = x_0$, when $\varpi_0 \approx \varpi^* = \gamma\sqrt{2}$. For a typical value for γ in the Earth's lower troposphere, $\varpi^* \approx 0.002$. We note that in geometric optics $\hat{y} = -(ncos\varphi/\gamma)^2$ and that the altitude $\hat{y} = \hat{y}_o = 0$ corresponds to a turning point. For $\varpi^* \approx 0.002$, it follows for the spherical case that when θ lies within ~0.1 deg of the turning point of a ray, the full Airy solution in Eq. (4.5-6) or its equivalent provides better accuracy. However, for angles of incidence in the range $0 \le \varphi <\sim \pi/2 - \cos^{-1}[\gamma\sqrt{2}]$, the asymptotic form for the characteristic matrix is adequate.

The TM case for constant gradient in the index of refraction follows in a similar way, but the equations are somewhat more complicated because of the presence of the $\nabla[\boldsymbol{E}\cdot(\nabla(\log\ \varepsilon)]$ term in the modified wave equation, which is absent in the TE case. However, for small gradients, it can be shown to first order in n' that the characteristic matrix elements for the TM case are given by

$$\left.\begin{aligned}
&F(\hat{y},\hat{y}_0)=\pi\frac{n}{n_0}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}, \quad f(\hat{y},\hat{y}_0)=i\pi\frac{nn_0}{\gamma}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\end{vmatrix},\\
&G(\hat{y},\hat{y}_0)=i\frac{\pi\gamma}{nn_0}\begin{vmatrix}\mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}, \quad g(\hat{y},\hat{y}_0)=-\pi\frac{n_0}{n}\begin{vmatrix}\mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\end{vmatrix},\\
&\hat{y}=-\varpi_{TM}^2\gamma^{-2}-\gamma(x-x_0), \qquad y(x_0)=\hat{y}_0
\end{aligned}\right\} \tag{4.5-14}$$

To convert these matrix elements into field components, Eqs. (4.2-7′) and (4.3-3) apply. For negative $\hat{y}$ values, it is readily shown that the asymptotic forms for the matrix elements in Eq. (4.5-14) approach to first order in $\varpi-\overline{\varpi}$ the values for the elements in Eq. (4.4-17′), which apply for the TM case.

4.6 Incoming and Outgoing Waves and Their Turning Points

Figure 4-2 shows ray paths for waves that are trending from left to right, and which are initially planar during their approach phase. In this example, we have two stratified layers infinite in extent parallel to the *yz*-plane with the index of refraction continuous across their boundary. The index of refraction is linearly varying in the lower medium, and it is constant in the upper medium. Across the boundary, n is continuous. The height $x = x_*$ marks the boundary between these two regimes; it is sufficiently far above the turning point height at $x = x_o$ (or, equivalently, with an angle of incidence sufficiently steep) so that asymptotic forms for the Airy functions can be applied to the characteristic

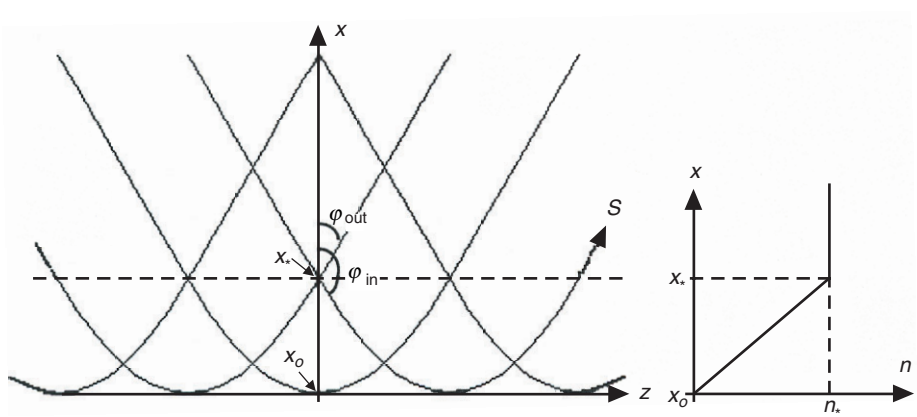

Fig. 4-2. Eikonal paths of incoming and outgoing waves in a two-layered Cartesian stratified medium. In the lower layer, the gradient of *n* is constant; in the upper layer, $n = n_* =$ constant.

matrix for heights at and above this boundary. Passing through any point above the turning point boundary at $x = x_o$ are both incoming and outgoing waves. The total field is given by the vector sum of these two waves.

For a planar TE wave impinging at an angle of incidence φ on a stratified surface in the homogeneous medium at or above the boundary $x = x_*$, the time-independent component of the electric field is given by

$$E = \exp\left(ikn_*\left(x\cos\varphi + z\sin\varphi + \text{constant}\right)\right) \tag{4.6-1}$$

Applying Maxwell's equations in Eq. (4.2-1a) to Eq. (4.6-1), one may express the angle of incidence in terms of the field components by

$$\tan\varphi = -\frac{H_x}{H_z} \tag{4.6-2}$$

For an incoming wave, H_x and H_z have the same polarity: they are either both positive or both negative (see Fig. 4-1). Therefore, it follows from Eq. (4.6-2) that $\pi/2 < \varphi \le \pi$. For an outgoing wave, H_x and H_z have opposite polarities, and it follows that $0 \le \varphi < \pi/2$.

At the boundary $x = x_*$, we have from Eqs. (4.2-8) and (4.2-9) for a planar wave

$$\tan\varphi_* = -\frac{W_*}{V_*} = n_o\frac{U_*}{V_*}, \quad \mu \equiv 1 \tag{4.6-3}$$

Using Snell's law in Eq. (4.2-14), it follows from Eq. (4.6-3) that

$$V_*^{\pm} = \pm n_* \,|\cos\varphi_*|\, U_*^{\pm} = \pm\gamma_*\sqrt{-\hat{y}_*}\,U_*^{\pm} \tag{4.6-4}$$

where the second relationship follows from Eq. (4.5-3). Here "+" denotes an outgoing wave where $0 \le \varphi < \pi/2$, and "–" denotes an incoming wave where $\pi/2 < \varphi \le \pi$.

When these constraining conditions between U_* and V_* are applied to the characteristic matrix solution for U and V given in Eq. (4.5-6), one obtains

$$\begin{pmatrix} U^{\pm} \\ V^{\pm} \end{pmatrix} = \pi\begin{pmatrix} \left(\mathrm{AiBi}_*' - \mathrm{Ai}_*'\mathrm{Bi}\right) \pm i\sqrt{-\hat{y}_*}\left(\mathrm{AiBi}_* - \mathrm{Ai}_*\mathrm{Bi}\right) \\ \gamma\left(i\left(\mathrm{Ai}'\mathrm{Bi}_*' - \mathrm{Ai}_*'\mathrm{Bi}'\right) \mp \sqrt{-\hat{y}_*}\left(\mathrm{Ai}'\mathrm{Bi}_* - \mathrm{Ai}_*\mathrm{Bi}'\right)\right) \end{pmatrix} U_*^{\pm} \tag{4.6-5}$$

Here $\mathrm{Ai} = \mathrm{Ai}[\hat{y}]$ and $\mathrm{Ai}_* = \mathrm{Ai}[\hat{y}_*]$; similarly for Bi. We now apply at the boundary $x = x_*$ the asymptotic forms in Eq. (3.8-4) for the Airy functions, which require that $\hat{y}_*$ be sufficiently negative. One obtains

$$\begin{pmatrix} U^{\pm} \\ V^{\pm} \end{pmatrix} \doteq \sqrt{\pi}\left(-\hat{y}_*\right)^{1/4} \begin{pmatrix} \mathrm{Ai}[\hat{y}] \mp i\,\mathrm{Bi}[\hat{y}] \\ \mathrm{i}\,\gamma\left(\mathrm{Ai}'[\hat{y}] \mp i\,\mathrm{Bi}'[\hat{y}]\right) \end{pmatrix} U_*^{\pm} \exp[\pm iX_*] \tag{4.6-6}$$

Here X_* is the phase accumulation along the x direction from the turning point at x_o up to x_* (plus $\pi/4$), which from Eq. (4.5-11) is given by

$$X_* = k\int_{x_o}^{x_*} \varpi dx + \frac{\pi}{4} \tag{4.6-7}$$

We note from Eq. (4.6-6) that the phases of $U^{\pm}$ and $V^{\pm}$ approach constant values for increasing $\hat{y} > 0$.

The total field at any point is given by adding the incoming and outgoing components. This sum must be devoid of the Airy function of the second kind, $\mathrm{Bi}[\hat{y}]$, in order to satisfy the physical "boundary condition" that there be a vanishing field below the turning point line, that is, in the region where $\hat{y}$ is positive; only $\mathrm{Ai}[\hat{y}]$ vanishes for positive $\hat{y}$. A turning point represents a kind of grazing reflection. From Eq. (4.6-6), we see that we can null the $\mathrm{Bi}[\hat{y}]$ component in the sum of the incoming and outgoing components if we set

$$U_*^{+} \exp[iX_*] = U_*^{-} \exp[-iX_*] \tag{4.6-8}$$

Thus, U_*^{+} equals U_*^{-} plus a phase delay Φ that is given by

$$\Phi = 2k\int_{x_o}^{x_*} \varpi dx + \frac{\pi}{2} \tag{4.6-9}$$

which is the round-trip phase delay along the x-direction (plus $\pi/2$).

Alternatively, from symmetry considerations it follows that U_*^{+} is the complex conjugate of U_*^{-} plus a correction arising from the variability of the index of refraction in the lower medium. It is easily shown from Eqs. (4.6-7) and (4.6-8) that if the incoming planar wave at the boundary has the form $U_*^{-} = \exp[ik\varpi_* x_* + \pi/4]$, then the outgoing planar wave at the boundary has the form

$$U_*^{+} = \exp\left[-i\left(k\varpi_* x_* + \frac{\pi}{4}\right)\right]\exp\left[i2k\int_{x_o}^{x_*} \varpi' x dx\right] \tag{4.6-10}$$

There is another way of interpreting incoming and outgoing waves using the characteristic matrix. From Eq. (4.5-6) and the Wronskian for the Airy functions, it follows that

$$\left.\begin{aligned} &\text{If}\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = C\begin{pmatrix} \mathrm{Ai}[\hat{y}_0] \\ i\gamma\,\mathrm{Ai}'[\hat{y}_0] \end{pmatrix}, \\ &\text{then}\begin{pmatrix} U \\ V \end{pmatrix} = \boldsymbol{M}[\hat{y},\hat{y}_0]\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = C\begin{pmatrix} \mathrm{Ai}[\hat{y}] \\ i\gamma\,\mathrm{Ai}'[\hat{y}] \end{pmatrix} \;\; \forall \;\; \hat{y} \end{aligned}\right\} \tag{4.6-11}$$

Here C is a constant. If we let $\hat{y}$ be sufficiently negative so that the asymptotic forms for the Airy functions apply, and we set $C = 2i\pi^{1/2}$, then it follows that

$$\left.\begin{aligned} &\begin{pmatrix} U \\ V \end{pmatrix} \to (-\hat{y})^{\frac{1}{4}}\begin{pmatrix} \exp(i\,\mathrm{X}) - \exp(-i\,\mathrm{X}) \\ \underset{\substack{\uparrow \\ \text{Outgoing}}}{\big(\exp(i\,\mathrm{X})} + \underset{\substack{\uparrow \\ \text{Incoming}}}{\exp(-i\,\mathrm{X})\big)} n\cos\varphi \end{pmatrix} \\ &\mathrm{X} = \frac{2}{3}(-\hat{y})^{3/2} + \frac{\pi}{4} \end{aligned}\right\} \tag{4.6-12}$$

Thus, U and V at the position $\hat{y}$ can be interpreted as consisting of the *superposition* of an incoming wave with a phase $-X(\hat{y}) - \pi$ and an angle of incidence $\pi - \varphi$, and an outgoing wave with a phase $X(\hat{y})$ and an angle of incidence φ.

Using Eq. (4.6-8), the total field at any point is given in terms of the incoming planar wave at the boundary by the expressions

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U^+ \\ V^+ \end{pmatrix} + \begin{pmatrix} U^- \\ V^- \end{pmatrix} = 2\sqrt{\pi}(-\hat{y}_*)^{1/4}\begin{pmatrix} -i\,\mathrm{Ai}[\hat{y}] \\ \gamma\,\mathrm{Ai}'[\hat{y}] \end{pmatrix} U_*^- \exp(+iX_*) \tag{4.6-13}$$

We note that V describes the tangential component of the magnetic field H_z in the TE case, that is, the component parallel to the plane of stratification. It must change sign near a turning point; in fact, in geometric optics it changes sign precisely at a turning point. Here this occurs when $\mathrm{Ai}'[\hat{y}] = 0$ for the last time before $\hat{y}$ becomes positive, which occurs at $\hat{y} = -1.02$. For heights below the turning point, that is, where $\hat{y}$ becomes positive, Eq. (4.6-13) shows that both U and V decay exponentially with increasing $\hat{y}$. In fact, these asymptotic forms for positive $\hat{y}$ are given in Eq. (3.8-4). They are

$$\left.\begin{aligned} \mathrm{Ai}[\hat{y}] &\to \frac{1}{2}\pi^{-1/2}\hat{y}^{-1/4}\exp\left(-\frac{2}{3}\hat{y}^{3/2}\right) \\ \mathrm{Ai}'[\hat{y}] &\to -\frac{1}{2}\pi^{-1/2}\hat{y}^{1/4}\exp\left(-\frac{2}{3}\hat{y}^{3/2}\right) \end{aligned}\right\} \tag{4.6-14}$$

Another way of viewing turning points is to consider the angle φ which the surface of constant phase (the cophasal surface) of the electric field for an incoming wave makes with the *yz*-plane. This angle is given by Eq. (4.2-13), where ψ is the phase of $U^{-}(y)$. From Eq. (4.6-6), it follows that the phase of the electric field for an incoming TE wave is given by

$$\psi = \tan^{-1}\left(\frac{\mathrm{Bi}[\hat{y}]}{\mathrm{Ai}[\hat{y}]}\right) - \mathrm{X}_* - kn_* x_* \cos\varphi_* \tag{4.6-15}$$

Using Eq. (4.5-3) for the relationship between $\hat{y}$ and x, and also the Wronskian for the Airy functions, it follows that

$$\frac{d\psi}{dx} = -\frac{k\gamma}{\pi\left(\mathrm{Ai}[\hat{y}]^2 + \mathrm{Bi}[\hat{y}]^2\right)} \tag{4.6-16}$$

Therefore, from Eq. (4.2-13), φ is given by

$$\tan\varphi = -\frac{\pi n_o}{\gamma}\left(\mathrm{Ai}^2[\hat{y}] + \mathrm{Bi}^2[\hat{y}]\right) = -\pi\tan\varphi_*\left(\mathrm{Ai}^2[\hat{y}] + \mathrm{Bi}^2[\hat{y}]\right)\left(-\hat{y}_*\right)^{1/2} \tag{4.6-17}$$

Thus, $\varphi \to \pi - \varphi_*$ as $\hat{y} \to \hat{y}_*$, as it must for an incoming plane wave with $\hat{y}$ sufficiently negative. Also, it follows that $\varphi \to \pi^+/2$ rapidly for increasing $\hat{y} > 0$.

From Eq. (4.6-6), we can evaluate the field components at any point, in particular at x_0 to obtain U_0 and V_0. Then we can write the field components evaluated at x in the form

$$\left.\begin{aligned} U^{\pm} &= \frac{\mathrm{Ai}[\hat{y}] \mp i\,\mathrm{Bi}[\hat{y}]}{\mathrm{Ai}[\hat{y}_0] \mp i\,\mathrm{Bi}[\hat{y}_0]} U_0^{\pm} \\ V^{\pm} &= \frac{\mathrm{Ai}'[\hat{y}] \mp i\,\mathrm{Bi}'[\hat{y}]}{\mathrm{Ai}'[\hat{y}_0] \mp i\,\mathrm{Bi}'[\hat{y}_0]} V_0^{\pm} \end{aligned}\right\} \tag{4.6-18}$$

We note the form $\mathrm{Ai}[\hat{y}] - i\,\mathrm{Bi}[\hat{y}]$ that is associated with an outgoing wave, and also the form $\mathrm{Ai}[\hat{y}] + i\,\mathrm{Bi}[\hat{y}]$ that is associated with an incoming wave. These are, of course, the same expressions contained in the asymptotic forms for the spherical Hankel functions [see Eq. (3.8-1)]. Thus, for spherical Hankel functions of the first kind, $\xi_l^+(\rho)/\rho$ is associated with outgoing waves because in the limit for large $\rho >> \nu$ it asymptotically approaches the form of an outgoing spherical wave. Similarly, for spherical Hankel functions of the

second kind, $\xi_l^-(\rho)/\rho$ approaches the form of an incoming wave, i.e., $\xi_l^\pm(\rho)/\rho \to i^{\pm(l+1)}\exp[\pm i\rho]/\rho,\ \rho >> l$. This formalism of identifying $\xi_l^+(\rho)$ with outgoing waves and $\xi_l^-(\rho)$ with incoming waves has already been used in Chapter 3 for scattering from a sphere. It was first pointed out by Hermann Hankel himself about 140 years ago.

4.6.1 Eikonal and Cophasal Normal Paths

In geometric optics, the optical path length $\mathscr{S}$ for a ray connecting two points is defined by

$$\mathscr{S} = \int n ds \tag{4.6-19}$$

where s is path length along a ray, and the integral is a path integral along the ray between the initial point and the end point. The path length vector infinitesimal $d\mathbf{s}$ at any point on the path is defined by

$$d\mathbf{s}/ds = \lim_{\lambda \to 0}[\mathbf{S}/S] \tag{4.6-20}$$

where $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{H})/4\pi n$ is the Poynting vector and λ is the wavelength of the electromagnetic wave. The Poynting vector is perpendicular to the wave front at any point, and its limiting form follows the path defined by the eikonal equation in geometric optics. (See [3] for a discussion of the foundations of geometric optics.)

$\mathscr{S}$ may be considered as a field quantity $\mathscr{S} = \mathscr{S}(x,y,z)$, which is associated with a family of ray paths passing through space. By varying the initial values of the rays, one generates a family of rays—for example, the family shown in Fig. 4-2. $\mathscr{S}$ is akin to the action integral in Hamilton-Jacobi theory or to the Feynman path integral in the sum-over-histories approach to quantum electrodynamics. $\mathscr{S}$ is a function only of the end point of the trajectory along which the path integral is evaluated. In geometric optics, it is the phase accumulated by following a Fermat path, that is, a path of stationary phase. Thus, the phase that would be accumulated by following any alternative path neighboring the Fermat path, but having the same end coordinates, assumes a stationary value $\mathscr{S}$ when the Fermat path is in fact followed. The evolution of $\mathscr{S}(x,y,z)$ as a field variable is governed by the eikonal equation[1]

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[1] The eikonal equation is related to the Hamilton-Jacobi partial differential equation, which arises in the Hamiltonian formulation of the Calculus of Variations problem for a Fermat path. In this formulation, a six-dimensional system of first-order ordinary differential equations in coordinate/conjugate momentum space determines a Fermat path in this six-dimensional space. The Hamilton-Jacobi equation describes the

$$|\nabla \mathscr{S}|^2 = n^2 \tag{4.6-21}$$

A surface $\mathscr{S}(x, y, z) = \text{constant}$ defines a wave front (in a geometric optics context), that is, a surface of constant phase across which a continuum of eikonal paths transect. The eikonal path through any point on the surface $\mathscr{S}(x, y, z) = \text{constant}$ is normal to it. The gradient $\nabla \mathscr{S} / |\nabla \mathscr{S}|$ is the unit tangent vector for an eikonal path. $\nabla \mathscr{S} / |\nabla \mathscr{S}|$ and the limiting form for the Poynting vector $\boldsymbol{S}$, as the wavelength of the wave approaches zero, are parallel; their magnitudes are related through a scale factor that equals the average electromagnetic power density of the wave.

For the case of Cartesian stratification with n' a constant, $\nabla \mathscr{S}$ is given by

$$\nabla \mathscr{S} = k^{-1}\left(\hat{\boldsymbol{x}} \ \frac{d\mathscr{A}}{dx} + \hat{\boldsymbol{z}} \frac{d\mathscr{C}}{dz} \right) = \hat{\boldsymbol{x}} \sqrt{n^2 - n_o^2} + \hat{\boldsymbol{z}} n_o \tag{4.6-22}$$

where $\mathscr{C}(z)$ is the phase accumulation of the time-independent component of the wave along the z-direction, that is, $\mathscr{C} = kn_o z$. $\mathscr{A}(x)$ is the phase accumulation along the x-direction, and it is given by Eq. (4.4-13). One can obtain $\mathscr{S}(x, y, z)$ from an integration of this gradient equation.

Note also that $\mathscr{S}(x, z)$ may not be unique, as is the case for the family of ray paths shown in Fig. 4-2. Since two ray paths pass through every point (x, z) above the altitude of the turning point, there are two functions, $\mathscr{S}^-(x, z)$ for the incoming path and $\mathscr{S}^+(x, z)$ for the outgoing path. Since the outgoing path has already touched the turning point line, which is a caustic surface in geometric optics and, therefore, an envelope to the system of ray paths, the outgoing path violates the Jacobi condition from the Calculus of Variations. This is a necessary condition that a stationary path must satisfy to provide a local minimum in the action integral, in this case the phase accumulation $\mathscr{S}$. In this example, $\mathscr{S}^+(x, z)$ provides a local maximum at the point (x, z).

For the incoming TE wave shown in Fig. 4-2, we can obtain the path generated by the normal to the cophasal surface of the electric field at any point, which essentially matches the eikonal path except near the turning point. In the plane of incidence, the coordinates for the normal path are given by its differential equation $dx / dz = \cot \varphi$, where φ is the angle of incidence and is given by Eq. (4.2-13). When the gradient of n is a constant, we obtain from Eqs. (4.5-4) and (4.6-17) for an Airy layer

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behavior of the stationary phase at the end point of the Fermat path over a region in this space that is spanned by a family of rays. The eikonal equation provides similar information in three-dimensional coordinate space for this family of rays.

$$\frac{dx}{dz} = -\gamma\left(\left(\mathrm{Ai}^2[\hat{y}] + \mathrm{Bi}^2[\hat{y}]\right)\pi n_o\right)^{-1} = -\gamma^{-1}k^{-1}\frac{d\hat{y}}{dz} \tag{4.6-23}$$

from which it follows that

$$\left.\begin{aligned} z &= \frac{\pi n_o}{k\gamma^2}\int\left(\mathrm{Ai}^2[\hat{y}] + \mathrm{Bi}^2[\hat{y}]\right)d\hat{y} + \text{constant} \\ &= \frac{\pi n_o}{k\gamma^2}\left(\left(\mathrm{Ai}^2[\hat{y}] + \mathrm{Bi}^2[\hat{y}]\right)\hat{y} - \left(\mathrm{Ai}'^2[\hat{y}] + \mathrm{Bi}'^2[\hat{y}]\right)\right) \\ &\xrightarrow[\hat{y}<<0]{} -\frac{2n_o}{k\gamma^2}\sqrt{-\hat{y}} \end{aligned}\right\} \tag{4.6-24}$$

Here we have chosen a particular normal path defined by the condition that $z = 0$ when $\hat{y} = \hat{y}^\dagger$, where $\hat{y}^\dagger$ is given by

$$\hat{y}^\dagger = \frac{\mathrm{Ai}'^2\left[\hat{y}^\dagger\right] + \mathrm{Bi}'^2\left[\hat{y}^\dagger\right]}{\mathrm{Ai}^2\left[\hat{y}^\dagger\right] + \mathrm{Bi}^2\left[\hat{y}^\dagger\right]} = 0.44133\cdots \tag{4.6-25}$$

It follows from Eqs. (4.5-3) and (4.6-24) that in the region where $\hat{y} << 0$, but in the lower layer, the normal path is given by

$$x - x_o = \frac{n'}{2n_o}z^2, \quad \hat{y} << 0 \tag{4.6-26}$$

From geometric optics, we have Snell's law $n\sin\theta = n_o$; it follows for an incoming wave that the eikonal path is given by

$$\frac{dx}{dz} = -\frac{\sqrt{n^2 - n_o^2}}{n} = -\frac{\sqrt{2n_o n'(x - x_o)}}{n_o + n'(x - x_o)} \tag{4.6-27}$$

Integrating yields

$$z = -2\sqrt{\frac{n_o(x - x_o)}{2n'}} - \frac{2}{3}\sqrt{\frac{n'(x - x_o)^3}{2n_o}} \tag{4.6-28}$$

When $n' << 1$, we can drop the second term, and this leads to essentially the same result given in Eq. (4.6-26) for the normal path for large negative $\hat{y}$. Only in the vicinity of the turning point (and below) does a significant divergence occur between an eikonal or ray path and the path generated by a normal to the cophasal surface of the electromagnetic wave.

Geometric optics is a second-order ray theory. The accuracy of geometric optics as a ray theory (and, therefore, as an approximate description of the cophasal normal path in wave theory) depends on the second variation of $\mathscr{S}(x,z)$ with respect to deviations from the nominal ray path being sufficiently large. This is equivalent to requiring the Fresnel approximation in stationary phase theory to be sufficiently accurate when applied to the nominal ray path. It can be shown in the example given in Fig. 4-2 that at $\hat{y} = \hat{y}_o = 0$ a caustic surface is generated where the second variation of $\mathscr{S}(x,z)$ is zero. A ray path having an interior contact point with a caustic surface usually is troublesome for second-order ray theory. Note that the equation in Eq. (4.6-24) forces z for the cophasal normal path to become very positive (for an incoming wave) when $\hat{y} > \hat{y}^{\dagger}$.

4.6.2 Defocusing

We also can calculate the "defocusing" caused by this refracting medium. We have represented the electromagnetic field of the harmonic wave by complex forms such as $U(x)\exp\left[i\left(kn_o z - \omega t\right)\right]$. However, we want a given physical property of these forms, such as electromagnetic energy, to be real. It is convenient for harmonic waves to use the complex Poynting vector [8] defined by $\boldsymbol{S} = c(\boldsymbol{E} \times \boldsymbol{H}^*)/8\pi$, where $\boldsymbol{H}^*$ is the complex conjugate of $\boldsymbol{H}$. The definition of $\boldsymbol{S}$ here includes a 1/2 term to reflect the root-mean-square energy flow of a harmonic wave over a cycle. The real part of $\boldsymbol{S}$, which is defined by $\mathrm{Re}[\boldsymbol{S}] = (\boldsymbol{S} + \boldsymbol{S}^*)/2$, gives the time-averaged flow of electromagnetic energy of the wave across a mathematical surface normal to $\boldsymbol{S}$. In Gaussian units, the dimensions of $\boldsymbol{E}$ and $\boldsymbol{H}$ are $(\text{mass}/\text{length})^{1/2}/\text{time}$. Therefore, $\boldsymbol{S}$ has the dimensions of power per unit area. At the boundary $x = x_*$ between the layers, this average energy flow for the incoming and outgoing planar waves is given by

$$8\pi\left\langle S_*^{\pm}\right\rangle = cn_*\left(\hat{z}\sin\theta_* \mp \hat{x}\cos\theta_*\right) \tag{4.6-29}$$

Here $\left|U_*^-\right|^2$, which has the dimensions of $\text{mass}/\left(\text{length}*\text{time}^2\right)$, has been set to unity. The energy flow of the superposition of the two waves is given by

$$8\pi\left\langle \mathrm{Re}\left[S_*^{\pm}\right]\right\rangle = \hat{z}cn_*\sin\theta_* \tag{4.6-30}$$

Thus, the time-averaged energy flow at $x = x_*$ is equal to the root-mean-square energy density of the wave times the component of its velocity along the z-axis.

The ratio R of the average energy flows at two different altitudes is given by

$$R = \frac{\langle \mathrm{Re}[\boldsymbol{S}_o] \rangle}{\langle \mathrm{Re}[\boldsymbol{S}_*] \rangle} = \frac{\mathrm{Ai}[\hat{y}_o]^2}{\mathrm{Ai}[\hat{y}_*]^2} \tag{4.6-31a}$$

At the altitude $\hat{y} = -1.02, V = i\gamma\,\mathrm{Ai}'[y] = 0$, which corresponds to a "turning point." At this point, we have

$$\langle R \rangle = 0.287\,\mathrm{Ai}[\hat{y}_*]^{-2} \underset{\hat{y}_* << 0}{\rightarrow} 1.8\sqrt{-\hat{y}_*} \rightarrow 1.8\gamma^{-1}\cos\theta_* \tag{4.6-31b}$$

This will be a large number for thin atmospheres except at near-grazing conditions. The square root of R yields the average voltage ratio of the electric fields at the two altitudes.

Figure 4-3 shows a family of eikonal paths generated from Eq. (4.6-24) by varying the angle of incidence φ_* at a given altitude x_*. The defocusing for a given path at a given point on the path is obtained by varying φ_*. To obtain the ratio of the displacement in x at the turning point and the perpendicular displacement of the path at another given point, one varies φ_* and calculates these two displacements. Their ratio gives the defocusing. The bold lines in Fig. 4-3 are the envelopes for this family of curves. These are caustic surfaces along which the defocusing is zero. A third-order or higher ray theory is required here to obtain the defocusing.

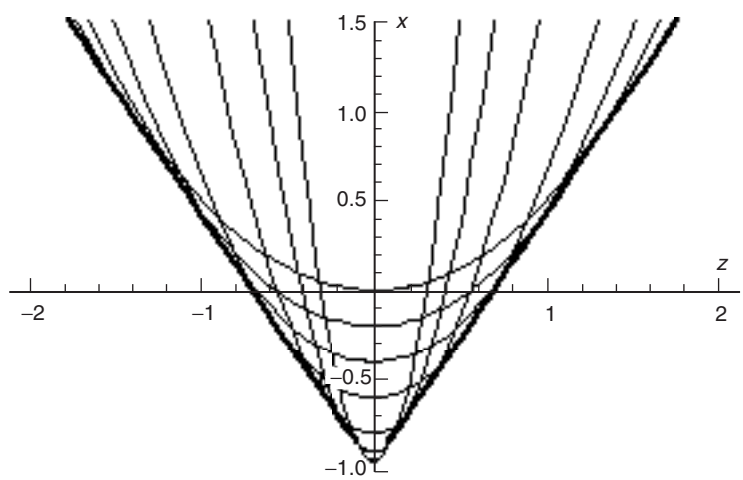

Fig. 4-3. Family of eikonal paths generated by varying the angle of incidence at a given altitude in an Airy layer.

4.7 Concatenated Airy Layers[2]

We now concatenate successive Cartesian layers; within a given layer the gradient of the index of refraction is held constant, but it is allowed to change from layer to layer. Because the elements of the characteristic matrix for each layer involve Airy functions, we call such a layer an "Airy layer." By varying n and n' across the boundaries between Airy layers and by allowing the thickness of the layers to shrink to zero while concomitantly allowing their number to grow infinitely large, we can create a discretely varying profile that matches a given continuously varying profile $n(x)$. In this approach, the gradient is stepwise constant, that is, it is constant within a layer but discontinuous across its boundary. In the former approach in Section 4.4, the gradient was zero but the index of refraction was stepwise constant. Using piecewise constant gradients in "onion skin" algorithms for recovery of atmospheric products from limb sounding data is considered to be more efficient than using zero gradient layers. This piecewise constant gradient approach may be useful near turning points.

Let us return to the wave equations for the Cartesian stratified case given in Eq. (4.2-10). We introduce the transformation

$$\left.\begin{aligned} \hat{y} &= -\gamma^{-2}\left(n^2 - n_o^2\right) \\ \gamma &= \left(2k^{-1}nn'\right)^{1/3} \end{aligned}\right\} \tag{4.7-1}$$

It follows that

$$\frac{d\hat{y}}{dx} = -k\gamma - \frac{2\hat{y}}{\gamma}\frac{d\gamma}{dx} \tag{4.7-2}$$

Within a given Airy layer, $\gamma_A(x)$ is a constant; hence, the wave equations in Eq. (4.5-4) and the Airy function formulation for the characteristic matrix given in Eq. (4.5-6) apply within that layer. The subscript "A" denotes the profile of the index of refraction and hence $\gamma_A(x)$ in the Airy layer. Within a layer, we match this piecewise constant function to the mean value of the actual profile $\gamma(x)$:

$$\gamma_A(x) = \frac{1}{2}\left(\gamma\left(x_{j-1}\right) + \gamma\left(x_j\right)\right), \quad x_{j-1} \le x < x_j \tag{4.7-3}$$

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[2] This section offers an alternate basis to the Mie scattering approach for obtaining the forms of the osculating parameters in a stratified medium. Because the Mie scattering approach has been followed in Chapter 5, this section is not essential.

It follows from Eq. (4.7-2) that $\hat{y}_A$ varies linearly within an Airy layer because $\gamma_A(x)$ is a constant, but $\hat{y}_A$ is discontinuous across the boundary between layers. Examples of these profiles are shown in Fig. 4-4.

Within the *j*th Airy layer, $\hat{y}_A$ is given by

$$\hat{y}_A(x) = -\left(\gamma_A^{-2}\left(n^2 - n_o^2\right)\right)\Big|_{x_{j-1}} - k\gamma_A\big|_{x_{j-1}}\left(x - x_{j-1}\right), \quad x_{j-1} \le x < x_j \tag{4.7-4a}$$

Across the upper boundary of the *j*th layer the discontinuity in $\hat{y}_A$ is given by

$$\Delta\hat{y}_A\big|_{x_j} = -\frac{2\hat{y}_A}{\gamma_A}\Delta\gamma_A\Big|_{x_j} \tag{4.7-4b}$$

where $\Delta\gamma_A$ is the discontinuity required to maintain close tracking by $\gamma_A(x)$ of the actual profile $\gamma(x)$. $\Delta\gamma_A$ depends primarily on the curvature in $n(x)$. As

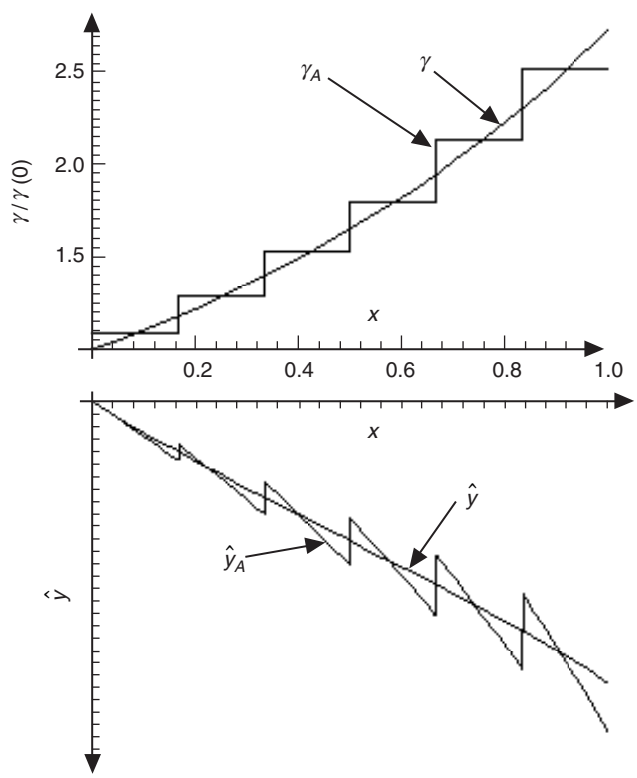

Fig. 4-4. Piecewise constant gradient profile for a series of Airy layers that approximate an exponential refractivity profile; $x = 1$ corresponds to 1 scale height above $x = 0$.

the maximum thickness of the Airy layers approaches zero and their number grows infinite, it follows that $\gamma_A(x) \to \gamma(x)$. From Eq. (4.7-4ab), it also follows that $\hat{y}_A$ will approach $\hat{y}(x)$ as defined by Eq. (4.7-1), but $\hat{y}'_A \neq \hat{y}'$. Accordingly, the characteristic matrix for the system of concatenated Airy layers spanning the space (x_0, x_N), and obtained from applying the product rule given in Eq. (4.4-2), will approach in the limit the characteristic matrix that applies to the actual medium with the profile $n(x)$.

In Cartesian stratification where n is held constant within a layer, Eq. (4.4-3) shows that sinusoidal functions result for the elements of the characteristic matrix. When the product rule in Eq. (4.4-2) is applied to obtain the reference characteristic matrix spanning multiple layers, Eq. (4.4-8) shows that sinusoids also result for the product. Here the argument of the sinusoids is the phase $\mathscr{A}$ accumulated along the x-axis. This "closed form" resulted from using the double angle formula, $\exp(iC) \times \exp(iD) = \exp[i(C+D)]$, to convert an infinite product of sinusoids into a single sinusoid with its argument containing an infinite sum, which can be represented by an integral. In this way, an infinite product of characteristic matrices is converted into a single characteristic matrix spanning all of the layers.

Unfortunately, no such "double angle" formula strictly applies to Airy functions. When $\hat{y}$ is sufficiently negative so that the sinusoidal asymptotic forms for the Airy functions can be used, then the double angle formula can be applied with sufficient accuracy. Near turning points where $\hat{y}$ is near zero, that is, where $x - x_o <\sim 2k^{-1}\gamma^{-1}$ (typically a few tens of meters for thin-atmosphere conditions with L-band signals), then the Airy functions themselves, or their equivalents, should be used. So, upon application of the product rule, we will need another strategy to convert an infinite product of characteristic matrices into a tractable form—in particular, into a single matrix spanning the entire sequence of layers.

The transitive property of characteristic matrices, $\boldsymbol{M}[\hat{y}_2, \hat{y}_1]\boldsymbol{M}[\hat{y}_1, \hat{y}_0] = \boldsymbol{M}[\hat{y}_2, \hat{y}_0]$, can be used even when the elements involve Airy functions. This is proved by using the Wronskian for the Airy functions. But this transitive property requires that the "$\hat{y}_1$" in $\boldsymbol{M}[\hat{y}_2, \hat{y}_1]$ have the same value as the "$\hat{y}_1$" in $\boldsymbol{M}[\hat{y}_1, \hat{y}_0]$. When one changes n' across the boundary $x = x_j$ between two Airy layers, Eq. (4.7-1) shows that $\hat{y}_A$ will change unless n_A and n'_A are concurrently changed to keep $\hat{y}_A$ invariant across the boundary. From Eq. (4.7-1), if we set $\delta\hat{y}_A = 0$, this requires the constraint

$$\left.\frac{\delta n'}{n'}\right|_{x_j} = \left(\frac{2n^2 + n_o^2}{n^2 - n_o^2}\right)\left.\frac{\delta n}{n}\right|_{x_j} \tag{4.7-5}$$

to hold between δn and $\delta n'$. It follows that

$$\left.\frac{\delta\gamma}{\gamma}\right|_{x_j} = \left.\left(\frac{n^2}{n^2 - n_o^2}\frac{\delta n}{n}\right)\right|_{x_j} = \left.\left(\frac{n^2}{2n^2 + n_o^2}\frac{\delta n'}{n'}\right)\right|_{x_j} \tag{4.7-6}$$

must hold across a boundary if $\hat{y}_A$ is to remain invariant.

Alternate strategies can be followed subject to the constraint in Eq. (4.7-5). Let us set a value for the change in the index of refraction across the boundary between Airy layers, Δn_A, so that at the layer boundary $x = x_j$ the value of the index of refraction for the piecewise constant gradient profile n_A matches the value $n(x_j)$ of the actual profile being modeled. One obtains a profile such as that shown in Fig. 4-5. In this scenario, it follows that at the jth boundary Δn_A is given by $\Delta n_A = \left(n'(x_{j-1}) - n'_{Aj}\right)(x_j - x_{j-1}) + \mathrm{O}\left[(x_j - x_{j-1})^2\right]$. Here n'_{Aj} is constant in the jth layer, and $\Delta n_A\big|_{x_j}$ is the required discontinuity at the jth boundary. Going to the limit, we obtain differential equations for $\hat{y}_A$ and γ_A that are given by

$$\left.\begin{aligned} &\frac{d\hat{y}_A}{dx} = -k\gamma_A, \quad \frac{1}{\gamma_A}\frac{d\gamma_A}{dx} = \frac{k}{2}\frac{\gamma^3 - \gamma_A^3}{n^2 - n_o^2}, \\ &\gamma_A^3 = 2k^{-1}n(x)n'_A(x), \quad \gamma^3 = 2k^{-1}n(x)n'(x) \end{aligned}\right\} \tag{4.7-7a}$$

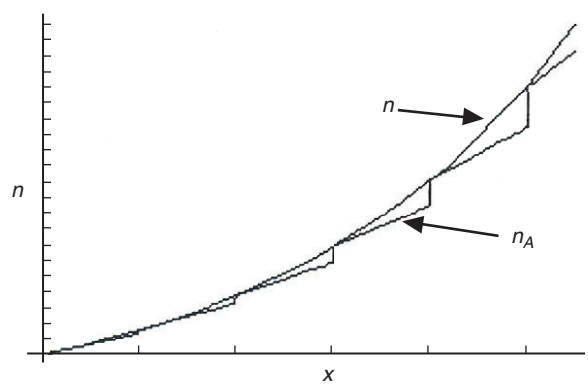

Fig. 4-5. Piecewise constant gradient profile for a series of Airy layers; $n_A(x)$ is matched to the actual profile $n(x)$, while keeping $\hat{y}$ continuous across the boundaries between layers.

It follows from Eqs. (4.7-1) and (4.7-7a) that

$$\hat{y}_A \gamma_A^2 = \hat{y}\gamma^2 \tag{4.7-7b}$$

The rate of phase accumulation along the x-axis is the same in the Airy layers as it is in the actual medium.

We are approximating the continuous index of refraction for the medium by an infinite stack of Airy layers. Suppose we set $\hat{y} = \hat{y}_0 = \hat{y}_o = 0$, and at this point we match the refractivity and its gradient to the actual profile that is being modeled, i.e., we equate $n_A(x_o) = n(x_o) = n_o$ and $n'_A(x_o) = n'(x_o)$; therefore, $\gamma_A(x_o) = \gamma(x_o)$. This provides boundary conditions for Eq. (4.7-7a) from which the profile for $\hat{y}_A$ and $d\hat{y}_A / dx$ can be obtained. In this manner, we can adjust n'_A in our piecewise constant gradient profile to best match the actual profile of the index of refraction. But it should be noted that $n'_A \neq n'(x)$, even in the limit; also, Eq. (4.7-7a) shows that $\gamma_A(x_j) \neq \gamma(x_j)$, except initially. Figure 4-6 shows an example of the differing profiles for γ_A and γ, in this case for an exponentially stratified medium. Their rate of divergence essentially depends only on the curvature in $n(x)$. For the exponential profile, it is independent of N_o; it depends only on H, the scale height.

For thin-atmosphere conditions, γ will be small and the difference $\gamma - \gamma_A$, which is zero at $\hat{y} = 0$, will be smaller. Accordingly, we let $\gamma_A = \gamma + w$, keeping only first-degree terms in w. A first-order differential equation for w truncated to first degree follows from Eq. (4.7-7a):

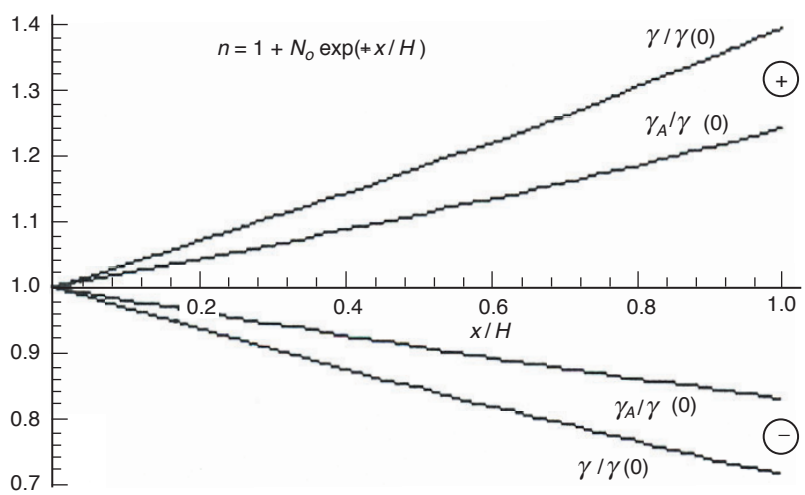

Fig. 4-6. Divergence between γ and γ_A from forcing $\hat{y}$ to be continuous across Airy layer boundaries.

$$\left.\begin{aligned}&\frac{dw}{dx}=-\frac{3w}{2}\frac{d}{dx}\log(|n^2-n_o^2|)-\frac{d\gamma}{dx};\ \ w(x_o)=0,\\&\gamma_A=\gamma+w,\ \ \gamma_A^3=2k^{-1}n(x)n_A'(x),\ \ \gamma^3=2k^{-1}n(x)n'(x)\end{aligned}\right\}\tag{4.7-8}$$

This differential equation has as a solution

$$w=-\left(n^2-n_o^2\right)^{-3/2}\int_{x_o}^{x}\left(n^2-n_o^2\right)^{3/2}\frac{d\gamma}{dx}dx\tag{4.7-9}$$

Given the actual profile $n(x)$, a first-order correction term $w(x)$ can be obtained from Eq. (4.7-9), and, therefore, the profile $\gamma_A(x)$ is obtained for the medium being modeled by Airy layers. It follows from Eqs. (4.7-7) and (4.7-9) that $w(x)$ depends primarily on the curvature in $n(x)$. The profile $\hat{y}_A(x)$ follows from Eqs. (4.7-1) and (4.7-7b), that is,

$$\hat{y}_A(x)=-k\int_{x_o}^{x}\gamma_A(x')dx'=\hat{y}(x)\left(\gamma(x)/\gamma_A(x)\right)^2\tag{4.7-10}$$

In an alternate strategy, one would set $n_A'(x_j)=n'(x_j)$ at each layer boundary with the initial conditions $n_A(x_o)=n(x_o)$. What strategy we should follow is dictated by the requirement that the results for the characteristic matrix from a given strategy match at least asymptotically the characteristic matrix from the first-order theory given in Section 3.5 for the case where n is piecewise constant profile. Without pinning down the strategy now, let us simply define $\gamma_A(x)$ as a piecewise constant function for a stack of Airy layers, and we will return to its form later.

With the profiles for $n_A(x_j)$ and $n_A'(x_j)$ that keep $\hat{y}$ continuous across the boundaries between Airy layers, one can develop a characteristic matrix for the stack. This is described in Appendix H.

4.8 Osculating Parameters

We return to a general Cartesian stratified medium (Fig. 4-1) for the TE case with $n=n(x)$. We have the coupled system

$$\left.\begin{aligned}&E_y=U(u),\ \ H_z=V(u),\ \ H_x=W(u),\\&\frac{dU}{du}=i\mu V,\ \ \frac{dV}{du}=i\varpi^2U,\ \ \ \mu W+n_oU=0,\\&\mu\varpi^2=n^2-n_o^2,\ \ n_o=n\sin\varphi=\text{constant}\end{aligned}\right\}\tag{4.8-1}$$

where $u=kx$. Here we set $\mu\equiv1$ throughout the medium, but ϖ is variable.

Now we introduce an osculating parameter by solving for the reflection and transmission coefficients across a boundary that bears a discontinuity in n. To develop a functional form for these parameters, we first use the continuity conditions from Maxwell's equations that apply to the field components across a boundary. We obtain the changes in the parameters that result from a change in the index of refraction across a planar boundary, which is embedded in an otherwise homogeneous medium. After obtaining the transmission and reflection coefficients that apply across a boundary, we will use a limiting procedure to obtain a continuous version for these parameters. The continuity conditions are first-order condition, whereas the coupled equations in Eq. (4.8-1) are a second-order system. Therefore, these expressions provide approximate solutions. We will ascertain the limits of their validity.

For the case where the boundary carries neither charge density nor current density, and with $\mu \equiv 1$ throughout the medium, Maxwell's equations require for the TE case that across the boundary $E_y^{(i)} + E_y^{(r)} = E_y^{(t)}$, $H_z^{(i)} + H_z^{(r)} = H_z^{(t)}$, and also that $H_x^{(i)} + H_x^{(r)} = H_x^{(t)}$. Here, the superscript (i) denotes the incident wave, (r) the reflected wave, and (t) the transmitted wave. Reviewing Eq. (4.8-1), we see that these conditions are equivalent to requiring that $U(u)$ and $V(u)$ be continuous across the boundary at $u = u_*$. In Fig. 4-7, we assume that n_1 in the lower medium is constant, and similarly that n_2 is constant in the upper medium. But across the boundary, the index of refraction changes by $\Delta n = n_2 - n_1$. From Eq. (4.4-3), the solution for the field components in each homogeneous medium is given by

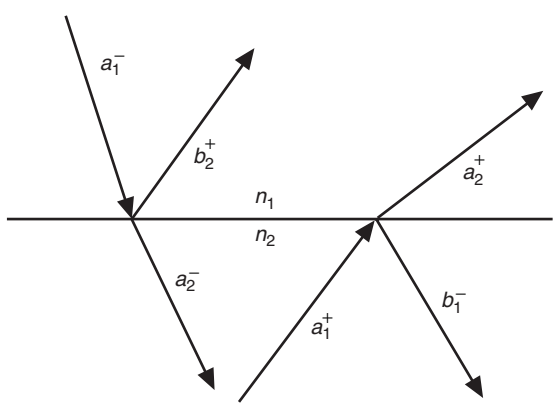

Fig. 4-7. Reflection and transmission coefficients across a boundary for upward and downward traveling waves.

$$\left.\begin{aligned}
&\tilde{U}^{\pm} = e^{\pm i\varpi u},\ \tilde{V}^{\pm} = \pm\varpi e^{\pm i\varpi u},\ \tilde{W}^{\pm} = -n_o e^{\pm i\varpi u},\\
&\varpi^2 = n^2 - n_o^2,\ n_o = n\sin\varphi = \text{constant},\\
&n = n_1,\ u < u_*;\quad n = n_2,\ u > u_*
\end{aligned}\right\} \tag{4.8-2}$$

In the solution $\tilde{U}^{\pm} = \exp(\pm i\varpi u)$, the plus sign is used for an upward-traveling wave, and the minus sign is used for a downward-traveling wave.

We write the solutions to Eq. (4.8-1) in terms of the basis functions $\tilde{U}^{\pm}$ and $\tilde{V}^{\pm}$ times an osculating parameter $a^{\pm}(u)$. Thus,

$$\left.\begin{aligned}
&U^{\pm} = a^{\pm}\tilde{U}^{\pm}\\
&V^{\pm} = -i\frac{dU^{\pm}}{du} = a^{\pm}\left(\varpi + u\varpi' - i\frac{a'^{\pm}}{a^{\pm}}\right)\tilde{U}^{\pm}\\
&(*)' = \frac{d(*)}{du}
\end{aligned}\right\} \tag{4.8-3}$$

We will show later that $(u\varpi' - a'/a)$ is small when evaluated away from a turning point and when thin-atmosphere conditions apply. For the case of a homogeneous medium, this term in Eq. (4.8-3) vanishes.

At a boundary, the incident radiation will split into a transmitted component and a reflected component. For an upward-traveling incident wave, the reflected component will be downward traveling and the transmitted component will be upward traveling. For each component, we have an osculating parameter, a for transmitted and b for reflected. The continuity conditions in this case of an upward traveling incident wave are

$$\left.\begin{aligned}
&a_1^+\tilde{U}_1^+ + b_1^-\tilde{U}_1^- = a_2^+\tilde{U}_2^+\\
&\varpi_1 a_1^+\tilde{U}_1^+ - \varpi_1 b_1^-\tilde{U}_1^- = \varpi_2 a_2^+\tilde{U}_2^+
\end{aligned}\right\} \tag{4.8-4}$$

Solving for a_2^+ and b_1^- in terms of a_1^+, one obtains

$$a_2^+ = \frac{2\varpi_1}{\varpi_1 + \varpi_2}\frac{1}{\tilde{U}_2^+\tilde{U}_1^-}a_1^+ \tag{4.8-5a}$$

$$b_1^- = \frac{\varpi_1 - \varpi_2}{\varpi_1 + \varpi_2}\frac{\tilde{U}_1^+}{\tilde{U}_1^-}a_1^+ \tag{4.8-5b}$$

which are essentially the Fresnel reflection and transmission formulas for the TE case.

For a series of layers, multiple internal reflections must be considered [9]. For example, a ray reflected downward from an upper -layer boundary will again be reflected upward at the boundary of interest. Figure 4-8 shows only rays that have been reflected twice to contribute to the upward-traveling main ray. Here N reflected rays, one from the boundary of each upward layer (the layers to the right of the jth layer in the figure), are then reflected again from the left-hand boundary of the jth layer. The reflection coefficients from these doubly reflected rays must be added to the transmission coefficient from main ray a_j^+ at the jth boundary to fully account for the total incident radiation at the left-hand boundary of the j + 1st layer. The second reflection also will occur from a lower boundary (to the left of the jth layer in Fig. 4-8), but reflections of this type will have already been folded into the value of a_j^+ at the left-hand boundary of the jth layer. However, Eq. (4.8-5b) shows that the reflection coefficients for doubly reflected rays will include a factor of the order of a_j^+ (here $\Delta\varpi$ is the average change in ϖ from layer to layer). Moreover, the phase of these secondary rays at the right-hand boundary of the jth layer will be randomly distributed when the span Δx of the ensemble of layers is such that $\Delta x >> \lambda$. It can be shown by vector summing up the contributions from all of these reflected rays with a second reflection from the left-hand boundary of the jth layer that the ratio of their combined contributions to the main ray contribution is given by $\lambda d\varpi / dx$, which is negligible for a thin atmosphere provided that turning points are avoided (where $\varpi = 0$ and $d\varpi / dx = nn' / \varpi \to \infty$). Therefore, in calculating the transmission coefficient for the wave, we can neglect secondary and higher-order reflections in our layer model when thin-atmosphere conditions apply and provided that we are sufficiently distant from a turning point.

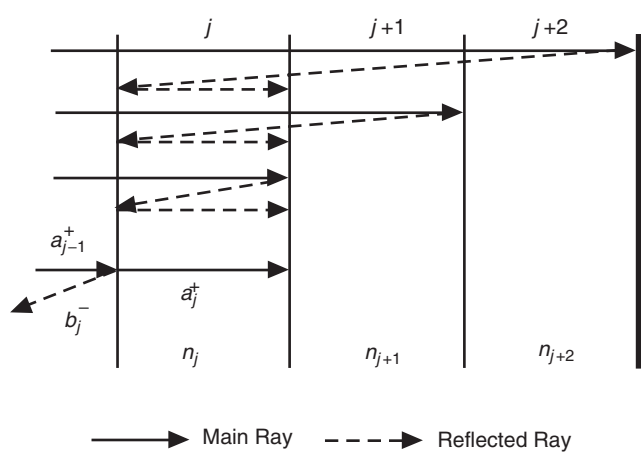

Fig. 4-8. Twice-reflected rays in the layered model.

The incident field at the j + 1st boundary can be considered as the *product* of the transmission coefficients from the previous j layers. If we then expand that product and retain only the first-order terms, we will obtain a first-order equation for the transmission coefficient. The range of validity of this linear truncation is essentially the same as that found for the truncation of the characteristic matrix to linear terms given in Section 4.4; i.e., thin-atmosphere conditions and away from turning points.

Let us define a_j^+ to be the transmission coefficient of an upward-traveling wave for the jth layer. Then, from Eq. (4.8-5a) it follows that

$$a_{j+1}^+ = \frac{2\varpi_j}{\varpi_j + \varpi_{j+1}} \frac{1}{\tilde{U}_j^- \tilde{U}_{j+1}^+} a_j^+ \tag{4.8-6}$$

It follows for a series of layers that

$$\left.\begin{aligned} a_{j+1}^+ &\doteq a_1^+ \left(\prod_{k=1}^{k=j} \left[\frac{2\varpi_k}{2\varpi_k + \Delta\varpi_k} \right] \right) \exp\left[-i \sum_{k=1}^{k=j} \Delta\varpi_k u_k \right] \\ \Delta\varpi_k &= \varpi_{k+1} - \varpi_k \end{aligned}\right\} \tag{4.8-7}$$

Equation (4.8-2) for $\tilde{U}^{\pm}(\varpi u)$ has been used to obtain the exponential term in Eq. (4.8-7). The product in Eq. (4.8-7) can be evaluated as

$$\begin{aligned} \log\left[\prod_{k=1}^{k=j} \left(\frac{2\varpi_k}{2\varpi_k + \Delta\varpi_k} \right) \right] &= \sum_{k=1}^{k=j} \log\left(\frac{2\varpi_k}{2\varpi_k + \Delta\varpi_k} \right) \\ &= -\sum_{k=1}^{k=j} \log\left(1 + \frac{\Delta\varpi_k}{2\varpi_k} \right) \doteq -\frac{1}{2} \sum_{k=1}^{k=j} \frac{\Delta\varpi_k}{\varpi_k} \end{aligned} \tag{4.8-8}$$

which is valid provided that $\Delta\varpi_k / \varpi_k$ is small. Going to the limit, we obtain from Eq. (4.8-7)

$$a^+(u) \doteq a^+(u_0) \sqrt{\frac{\varpi_0}{\varpi}} \exp\left(-i \int_{\varpi_0}^{\varpi} u d\varpi \right) \tag{4.8-9}$$

From Eq. (4.8-4), it follows for an upward-traveling wave that

$$U^+ = a^+ \tilde{U}^+ \doteq a_0^+ \sqrt{\frac{\varpi_0}{\varpi}} \exp\left(-i \int_{\varpi_0}^{\varpi} u d\varpi - \varpi u \right) \tag{4.8-10}$$

Integrating by parts yields the WKB solution (for this special case of a Cartesian stratified medium):

$$U_{\text{WKB}}^{+} = U_0^{+}\sqrt{\frac{\varpi_0}{\varpi}}\exp\left(i\int_{u_0}^{u}\varpi du\right) \tag{4.8-11}$$

Substituting the WKB solution in Eq. (4.8-11) into Eq. (4.8-1), one obtains

$$\left.\begin{aligned} \frac{d^2U_{\text{WKB}}^{+}}{du^2} &= \left(\frac{3}{4}\left(\frac{\varpi'}{\varpi}\right)^2 - \frac{\varpi''}{2\varpi} - \varpi^2\right)U_{\text{WKB}}^{+} \\ V_{\text{WKB}}^{+} &= -i\frac{dU_{\text{WKB}}^{+}}{du} = \left(\varpi + i\frac{\varpi'}{2\varpi}\right)U_{\text{WKB}}^{+} \end{aligned}\right\} \tag{4.8-12}$$

Thus, these solutions essentially satisfy the wave equations when $|\varpi'/\varpi| << 1$ and $|\varpi''/\varpi| << 1$, which, except at turning points, holds for thin-atmosphere conditions. Note that $\varpi'/\varpi = (n'/n)\sec^2\varphi$, where φ is the angle of incidence of the wave with the x-axis. At a turning point, $\varphi = \pi/2$.

For a downward wave, we define a_j^- to be the transmission coefficient of a downward-traveling wave for the *j*th layer. Applying the continuity conditions from Maxwell's equations at the boundary, and following the same arguments that led to Eq. (4.8-5a), one obtains

$$a_{j-1}^- = \frac{2\varpi_j}{\varpi_j + \varpi_{j-1}}\frac{1}{\tilde{U}_j^+\tilde{U}_{j-1}^-}a_j^- \tag{4.8-13}$$

Following the same procedures in Eq. (4.8-6) to Eq. (4.8-9) one obtains

$$a^-(u) = a^-(u_0)\sqrt{\frac{\varpi_0}{\varpi}}\exp\left(i\int_{\varpi_0}^{\varpi} u d\varpi\right) \tag{4.8-14}$$

The downward wave is given by the WKB solution:

$$U_{\text{WKB}}^{-} = a^-\tilde{U}^- = U_0^-\sqrt{\frac{\varpi_0}{\varpi}}\exp\left(-i\int_{u_0}^{u}\varpi du\right) \tag{4.8-15}$$

These are essentially the same forms given in the characteristic matrix in Section 3.5. From Eq. (4.8-9), we see that the phasor part of $a_j^{\pm}$ is just the phase accumulation resulting from a changing index of refraction between u_0 and u, i.e., $\varpi - \varpi_0$.

A further discussion of the higher-order correction terms from multiple reflections, including back-reflected waves, is given in [9].

4.8.1 At a Turning Point

At a turning point, these WKB solutions in Eqs. (4.8-11) and (4.8-15) fail. But we can approximate the turning point solution by use of the Airy solutions given in Section 4.5 when n' is a constant. Let us approximate the actual index of refraction in the vicinity of a turning point by $n = n_o + n_o'(u - u_o)$, where $n_o' = dn/du$ is evaluated at the turning point. It is assumed to remain constant in the vicinity of the turning points. The quadratic term in $n_o'(u - u_o)$ is negligible under our assumptions. Then the solution to the wave equations in the vicinity of the turning point is given by

$$\left.\begin{aligned} &U_A^{\pm} = K_1^{\pm}(\mathrm{Ai}[y] \mp i\,\mathrm{Bi}[y]), \quad V_A^{\pm} = i\gamma_o K_1^{\pm}(\mathrm{Ai}'[y] \mp i\,\mathrm{Bi}'[y]), \\ &\gamma_o = (2n_o n_o')^{1/3}, \;\; y = -\left(\frac{\varpi}{\gamma_o}\right)^2, \;\; \varpi^2 = n^2 - n_o^2 = \gamma_o^3(u - u_o) \end{aligned}\right\} \tag{4.8-16}$$

Here $u = kx$. At the turning point, $n = n_o$ and $y = 0$. Also, to ensure a vanishing field below the turning point, we must have $K_1^+ = K_1^- = K_1$, so that $U_A^+ + U_A^-$ involves only the Airy function of the first kind, $\mathrm{Ai}[y]$. At a point close to the turning point, but sufficiently above it so that $y <\sim -3$ (i.e., $u - u_o > 3/\gamma_o$, the equivalent of a few dekameters for the Earth's dry atmosphere at sea level), we may approximate the Airy functions with their negative argument asymptotic forms given in Eq. (3.8-7). The solutions in Eq. (4.8-16) become

$$U_A^{\pm} \doteq (-y)^{-1/4} K_1^{\pm} \exp\left[\pm i\left(\frac{2}{3}(-y)^{3/2} - \frac{\pi}{4}\right)\right] \tag{4.8-17}$$

But we recall from Section 4.5 that

$$\frac{2}{3}(-y)^{3/2} = \int_{u_o}^{u} \varpi\, du \tag{4.8-18}$$

Thus, Eq. (4.8-17) becomes

$$U_A^{\pm} \doteq K_1^{\pm} \sqrt{\frac{\mp i\gamma_o}{\varpi}} \exp\left(\pm i \int_{u_o}^{u} \varpi\, du\right) \tag{4.8-19}$$

The WKB solution at this point can be written in the form

$$U_{\mathrm{WKB}}^{\pm} = K_2^{\pm} \varpi^{-1/2} \exp\left(\pm i \int_{u_o}^{u} \varpi\, du\right) \tag{4.8-20}$$

Matching the coefficient in the Airy function asymptotic form with the coefficient in the WKB solution, we have

$$C_1^{\pm} \Leftrightarrow C_2^{\pm}\sqrt{\mp i\gamma_o} \tag{4.8-21}$$

The coefficients $C_2^{\pm}$ come from matching the WKB solutions with the boundary conditions for the problem, for example, those resulting from matching the solution to a plane incident or exiting wave at a particular altitude (see Fig. 4-2) above the turning point. With $C_1^{\pm}$ determined from Eq. (4.8-21), we then have an approximate solution in Eq. (4.8-16) valid at a turning point expressed in terms of Airy functions.

4.9 Airy Functions as Basis Functions

We can carry the use of osculating parameters a step further by defining the basis functions by

$$\left.\begin{aligned} &\tilde{U}_A^{\pm} = K_1^{\pm}\left(\mathrm{Ai}[y] \mp i\,\mathrm{Bi}[y]\right), \quad \tilde{V}_A^{\pm} = i\gamma_o K_1^{\pm}\left(\mathrm{Ai}'[y] \mp i\,\mathrm{Bi}'[y]\right), \\ &\gamma_o = \left(2n_o n_o'\right)^{1/3}, \quad y = -\left(\frac{\varpi}{\gamma_o}\right)^2, \quad \varpi^2 = n^2 - n_o^2 = \gamma_o^3\left(u - u_o\right) \end{aligned}\right\} \tag{4.9-1}$$

which are the exact solutions when $n' = dn/du$ is a constant in the medium and the quadratic term in n^2 can be ignored. When n' is variable, then γ is variable.

For the TE case, we recall that both $\tilde{U}$ and $\tilde{V}$ must be continuous across a boundary. At a boundary where n' changes discontinuously, the incident radiation will split into a transmitted component and a reflected component. For each component, we have an osculating parameter, a for transmitted and b for reflected. Therefore, the continuity conditions require

$$\left.\begin{aligned} a_1^{\pm}\tilde{U}_1^{\pm} + b_1^{\mp}\tilde{U}_1^{\mp} &= a_2^{\pm}\tilde{U}_2^{\pm} \\ i\gamma_1 a_1^{\pm}\tilde{U}_1'^{\pm} - i\gamma_1 b_1^{\mp}\tilde{U}_1'^{\mp} &= i\gamma_2 a_2^{\pm}\tilde{U}_2'^{\pm} \end{aligned}\right\} \tag{4.9-2}$$

Solving this system for $a_2^{\pm}$ and $b_1^{\mp}$ in terms of $a_1^{\pm}$, we obtain

$$\left.\begin{aligned} D^{\pm}a_2^{\pm} &= 2\gamma_1\pi^{-1}a_1^{\pm} \\ D^{\pm}b_1^{\mp} &= i\left(\gamma_1\tilde{U}_2^{\mp}\tilde{U}_1'^{\mp} - \gamma_2\tilde{U}_1^{\mp}\tilde{U}_2'^{\mp}\right)a_1^{\pm} \\ D^{\pm} &= \pm i\left(\gamma_1\tilde{U}_2^{\pm}\tilde{U}_1'^{\mp} + \gamma_2\tilde{U}_1^{\mp}\tilde{U}_2'^{\pm}\right) \end{aligned}\right\} \tag{4.9-3}$$

Now we let $\gamma_1 = \gamma - \Delta\gamma/2$, $\gamma_2 = \gamma + \Delta\gamma/2$, and $\Delta y = -2y\Delta\gamma/\gamma$. We expand $D^{\pm}$ in Eq. (4.9-3) to first order in $\Delta\gamma/\gamma$, using Eq. (4.9-1). We obtain

$$D^{\pm} = \pm\frac{2\gamma}{\pi}\left(1 + i\frac{\pi}{2}\frac{\Delta\gamma}{\gamma}\Gamma\right) \tag{4.9-4}$$

where $\Gamma[\hat{y}]$ is given by

$$\begin{aligned}\Gamma = &\left(\mathrm{Ai}[y]\mathrm{Ai}'[y] + \mathrm{Bi}[y]\mathrm{Bi}'[y]\right)\\ &+ 2y\left(\mathrm{Ai}'^2[y] + \mathrm{Bi}'^2[y] - y\left(\mathrm{Ai}^2[y] + \mathrm{Bi}^2[y]\right)\right)\end{aligned} \tag{4.9-5}$$

The incident field at the j + 1st boundary can be expressed as the product of the transmission coefficients from the previous j layers. We note that $\Delta\gamma/\gamma = (1/3)\Delta n'/n'$. Thus, the transmission coefficients can be written as

$$a_{j+1}^{\pm} = a_1^{\pm}\prod_{k=1}^{k=j}\left(2\gamma_k\left(\pi\,|\,D_k^{\pm}\,|\right)^{\mp 1}\right) \tag{4.9-6}$$

Going to the limit, we obtain

$$\left.\begin{aligned}&a^{\pm} = a_0^{\pm}\sqrt{\frac{\gamma_0}{\gamma}}\exp\left[\mp i\frac{\pi}{6}\left(\int_{u_0}^{u}\frac{d\log n'}{du}\Gamma[y]du\right)\right],\\ &y = -(\varpi/\gamma)^2,\ \gamma = (2nn')^{1/3},\ \varpi^2 = n^2 - n_o^2\end{aligned}\right\} \tag{4.9-7}$$

Therefore, this form for the osculating parameters should be good up to a turning point, provided $n' \neq 0$. It follows from Eq. (4.9-5) that $d\Gamma/dy = 0$ at $y = y^{\dagger} = 0.44331$. For $y > y^{\dagger}$, $\Gamma[y]$ grows large rapidly because it contains Airy functions of the second kind. This approach fails for increasing $y > y^{\dagger}$.

4.10 Wave Propagation in a Cylindrical Stratified Medium

We develop the characteristic matrix for cylindrical stratification. We again consider the TE case in two dimensions, that is, in the *xz*-plane in Fig. 4-9. The field is assumed to be invariant in the *y*-direction, along the axis of the cylinder. For the TE wave in this case, $E_r = E_\theta = 0$ and $H_y = 0$. When μ and ε are constant, Bessel functions of integer order l are solutions for the radial coordinate of the electromagnetic field. The mathematics is simpler for the cylindrical case than it is for the spherical case, but it is a simple extension to convert to spherical Bessel functions, once the cylindrical case is determined.

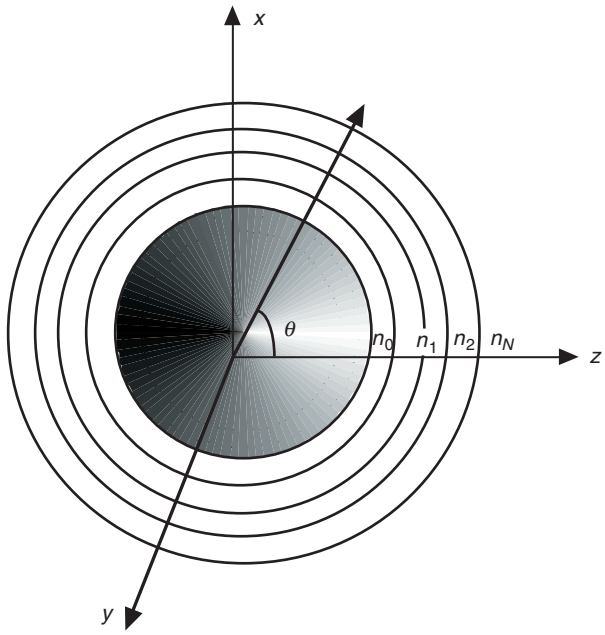

Fig. 4-9. Cylindrical stratified index of refraction.

Spherical stratification will be used to tie these results with those that follow from the spherical scattering coefficient approach, which is discussed later.

For a TE wave in the xz-plane ($\boldsymbol{E}$ along the y-axis), Maxwell's curl equations in Eq. (3.2-1a) expressed in cylindrical coordinates become

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial E_y}{\partial \theta} &= ik\mu H_r \\ -\frac{\partial E_y}{\partial r} &= ik\mu H_\theta \\ 0 &= H_y \end{aligned}\right\} \tag{4.10-1a}$$

$$\left.\begin{aligned} 0 &= E_r \\ 0 &= E_\theta \\ \frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta}\right) &= -ik\varepsilon E_y \end{aligned}\right\} \tag{4.10-1b}$$

We now use the separation-of-variables technique to solve these equations. We see from the second equation in Eq. (4.10-1a) that for the θ coordinate H_θ and

E_y must be represented by the same function times at most a constant. Accordingly, we set

$$\left.\begin{aligned} E_y &= U(r)\Phi(\theta) \\ H_\theta &= V(r)\Phi(\theta) \\ H_r &= W(r)\Theta(\theta) \end{aligned}\right\} \tag{4.10-2}$$

Inserting these forms into Eq. (4.10-1), it follows that conditions for these representations to be valid for all values of r and θ are that $\Phi' = a\Theta$ and $\Theta' = b\Phi$, where a and b are constants. Also, to ensure single-valuedness in these θ functions, the product ab must equal $-l^2$, where l must be integer-valued. Thus, both Φ and Θ are of the form $\exp(il\theta)$. It follows that $a = b = \pm il$. Separating the variables in Eq. (4.10-1) requires that the following system of equations must hold:

$$\mu \frac{d}{dr}\left(\frac{1}{\mu} r \frac{dU}{dr}\right) + \left(k^2 n^2 - \frac{l^2}{r^2}\right) rU = 0 \tag{4.10-3a}$$

$$\left.\begin{aligned} \frac{1}{knr}\frac{d(rV)}{dr} &= -i\sqrt{\frac{\varepsilon}{\mu}}\left(1 - \frac{l^2}{(knr)^2}\right)U \\ \frac{1}{kn}\frac{dU}{dr} &= -i\sqrt{\frac{\mu}{\varepsilon}}V \end{aligned}\right\} \tag{4.10-3b}$$

$$nkrW = l\sqrt{\frac{\varepsilon}{\mu}}U \tag{4.10-3c}$$

$$\frac{d^2\Theta}{d\theta^2} = \frac{d^2\Phi}{d\theta^2} = -il^2\Theta \tag{4.10-3d}$$

Thus, the eigenfunction solutions to Eq. (4.10-3d), irrespective of the functional form of $n(r)$, are given by

$$\Theta_l = \Phi_l = \exp(il\theta), \;\; l = 0, \pm 1, \pm 2, \cdots \tag{4.10-3d$'$}$$

The complete solution will be a weighted summation of the individual spectral components, $U_l\Phi_l$, $V_l\Phi_l$, and $W_l\Theta_l$, over all integer values of l, which for each value of l are solutions to Eq. (4.10-3). For the special case of $l = 0$, we see from Eqs. (4.10-1) and (4.10-2) that $\Phi_0 \equiv 1$ and that $W_0 \equiv 0$. U_0 and V_0 are given by Eqs. (4.10-3a) and (4.10-3b) with $l = 0$.

Equation (4.10-3a) provides the modified wave equation for $U_l(r)$, which describes the lth spectral component of the electric field for a TE wave. It is the radial part of the modified vector wave equation given in Eq. (3.2-1b). In a medium with $\mu \equiv 1$, this wave equation in Eq. (4.10-3a) simplifies further to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU_l}{dr}\right)+\left(k^2n^2-\frac{l^2}{r^2}\right)U_l=0 \qquad (4.10\text{-}3a')$$

which is the modified Bessel equation discussed in Section 5.2.

When μ and ε are constant, one obtains from Eq. (4.10-3) as spectral solutions

$$\left.\begin{aligned} &U_l(\rho)=\{J_l(\rho),Y_l(\rho)\},\ V_l(\rho)=i\sqrt{\frac{\varepsilon}{\mu}}\{J_l'(\rho),Y_l'(\rho)\},\quad \rho=nkr \\ &\rho W_l(\rho)=l\sqrt{\varepsilon/\mu}U_l(\rho),\ \Theta_l(\theta)=\exp(il\theta),\quad l=0,\pm1,\pm2,\cdots \end{aligned}\right\} \qquad (4.10\text{-}4)$$

where $J_l(\rho)$ and $Y_l(\rho)$ are the Bessel functions of the first and second kind, discussed earlier.

If we write in characteristic matrix form the solutions to the coupled system in Eq. (4.10-3b), we obtain

$$\begin{pmatrix} U_l(\rho) \\ \rho V_l(\rho) \end{pmatrix}=\boldsymbol{M}_l[\rho,\rho_0]\begin{pmatrix} U_l(\rho_0) \\ \rho_0 V_l(\rho_0) \end{pmatrix}=\begin{bmatrix} F_l(\rho,\rho_0) & f_l(\rho,\rho_0) \\ G_l(\rho,\rho_0) & g_l(\rho,\rho_0) \end{bmatrix}\begin{pmatrix} U_l(\rho_0) \\ \rho_0 V_l(\rho_0) \end{pmatrix} \qquad (4.10\text{-}5)$$

One can show that the elements of the characteristic matrix for the TE case, which are solutions to the coupled system of equations in Eq. (4.10-3b), are given by

$$\left.\begin{aligned} &F_l(\rho,\rho_0)=\frac{\pi\rho_0}{2}\begin{vmatrix} J_l(\rho) & Y_l(\rho) \\ J_l'(\rho_0) & Y_l'(\rho_0) \end{vmatrix},\ f_l(\rho,\rho_0)=i\frac{\pi}{2}\sqrt{\frac{\mu}{\varepsilon}}\begin{vmatrix} J_l(\rho) & Y_l(\rho) \\ J_l(\rho_0) & Y_l(\rho_0) \end{vmatrix}, \\ &G_l(\rho,\rho_0)=i\frac{\pi\rho\rho_0}{2}\sqrt{\frac{\varepsilon}{\mu}}\begin{vmatrix} J_l'(\rho) & Y_l'(\rho) \\ J_l'(\rho_0) & Y_l'(\rho_0) \end{vmatrix},\ g_l(\rho,\rho_0)=-\frac{\pi\rho}{2}\begin{vmatrix} J_l'(\rho) & Y_l'(\rho) \\ J_l(\rho_0) & Y_l(\rho_0) \end{vmatrix} \end{aligned}\right\} \qquad (4.10\text{-}6)$$

Here $J_l(\rho)Y_l'(\rho)-J_l'(\rho)Y_l(\rho)=(\pi\rho/2)^{-1}$, which is the Wronskian of the Bessel functions, has been used to set the scale factors so that $F_l(\rho,\rho_0)=1$ and $g_l(\rho,\rho_0)=1$. It is easily shown that when the elements of $\boldsymbol{M}_l[\rho,\rho_0]$ are given by Eq. (4.10-6), then $\mathrm{Det}[\boldsymbol{M}_l[\rho,\rho_0]]=1$ for all values of ρ.

For the TM case, starting from Maxwell's curl equations in Eqs. 4.2-1(a) and 4.2-1(b with $H_r = H_\theta = 0$ and $E_y = 0$, we use the separation of variables technique and set

$$\left.\begin{aligned} H_y &= U(r)\Phi(\theta) \\ E_\theta &= V(r)\Phi(\theta) \\ E_r &= W(r)\Theta(\theta) \end{aligned}\right\} \tag{4.10-7}$$

One can readily show for a TM wave that the equation set analogous to Eq. (4.10-3) is given by

$$\frac{\varepsilon}{r}\frac{d}{dr}\left(\frac{r}{\varepsilon}\frac{dU_l}{dr}\right) + \left(k^2n^2 - \frac{l^2}{r^2}\right)U_l = 0 \tag{4.10-8a}$$

$$\left.\begin{aligned} \frac{1}{knr}\frac{d(rV)}{dr} &= -i\sqrt{\frac{\varepsilon}{\mu}}\left(1 - \frac{l^2}{(knr)^2}\right)U \\ \frac{1}{kn}\frac{dU}{dr} &= -i\sqrt{\frac{\mu}{\varepsilon}}V \end{aligned}\right\} \tag{4.10-8b}$$

$$W_l = \frac{-l}{nkr}\sqrt{\frac{\mu}{\varepsilon}}U_l \tag{4.10-8c}$$

$$\Theta_l = \Phi_l = \exp(il\theta),\ l = 0. \pm 1, \pm 2, \cdots \tag{4.10-8d}$$

Equation (4.10-8a) shows for the TM case that a medium with $\mu \equiv 1$ does not offer further simplification of the modified wave equation for $U_l(r)$; not only is n variable, but we have the extra term involving $\nabla\left[\boldsymbol{E}\cdot(\nabla(\log\ \varepsilon)\right]$. This term can be avoided through the renormalization of n; see Appendix I.

For the TM case with constant n, the elements of the characteristic matrix have the same forms given in Eq. (4.10-6) except that the reciprocals of $\sqrt{\mu/\varepsilon}$ and $\sqrt{\varepsilon/\mu}$ are used in f and G_l.

The complete solution for either a TE wave or a TM wave is obtained by summing these spectral components weighted by the appropriate spectral coefficients on the integer spectral number l. The spectral coefficients might be obtained from asymptotic boundary conditions on $U(\rho_0, \rho_0)$ and $V(\rho_0, \rho_0)$, such as those for an approaching plane wave located at a large distance from the cylinder. Bauer's identity for a plane wave in cylindrical coordinates is given by

$$\exp(i\rho\cos\theta) = \sum_{l=-\infty}^{l=\infty} i^l J_l(\rho)\exp(il\theta) \tag{4.10-9}$$

For a TE wave, Eq. (4.10-9) provides the asymptotic form for E_y / E_o. Differentiating with respect to ρ and θ yields

$$\left.\begin{aligned} \exp(i\rho\cos\theta)\cos\theta &= \sum_{l=-\infty}^{l=\infty} i^{l-1} J_l'(\rho)\exp(il\theta) \\ \exp(i\rho\cos\theta)\sin\theta &= -\sum_{l=-\infty}^{l=\infty} i^l \frac{l}{\rho} J_l(\rho)\exp(il\theta) \end{aligned}\right\} \tag{4.10-10}$$

For a TE plane wave traveling in the positive z-direction in Fig. 4-9, these are the forms for $-H_\theta / H_o$ and $-H_r / H_o$, respectively. These forms also are obtained from Maxwell's equation $ik\mu\boldsymbol{H} = \nabla\times\boldsymbol{E}$ when $\boldsymbol{E} = \hat{\boldsymbol{y}}E_o\exp(i\rho\cos\theta)$. According to Eqs. (4.10-2) and (4.10-4), we set

$$\left.\begin{aligned} E_y &= \sum_{l=-\infty}^{l=\infty}(a_l J_l + b_l Y_l)\Theta_l = E_o\exp(i\rho\cos\theta) \\ H_\theta &= i\sqrt{\frac{\varepsilon}{\mu}}\sum_{l=-\infty}^{l=\infty}(c_l J_l' + d_l Y_l')\Theta_l = -H_o\exp(i\rho\cos\theta)\cos\theta \\ H_r &= \sqrt{\frac{\varepsilon}{\mu}}\sum_{l=-\infty}^{l=\infty}\frac{l}{\rho}(a_l J_l + b_l Y_l)\Theta_l = -H_o\exp(i\rho\cos\theta)\sin\theta \end{aligned}\right\} \tag{4.10-11}$$

Equating these forms to their corresponding forms in Eqs. (4.10-9) and (4.10-10), we obtain for the coefficients

$$\left.\begin{aligned} &a_l = E_o i^l,\ \ b_l \equiv 0 \\ &c_l = i^l H_o\sqrt{\mu/\varepsilon} = E_o i^l \\ &d_l \equiv 0 \end{aligned}\right\} \tag{4.10-12}$$

Similarly, suppose that the boundary conditions correspond to a plane wave along the line $\theta = -\alpha$. Then the time-independent form for the wave is given by $E_y = E_o\exp[i\rho\cos(\theta+\alpha)]$. It follows that Eq. (4.10-9) is modified to

$$\exp(i\rho\cos(\theta+\alpha)) = \sum_{l=-\infty}^{l=\infty} i^l J_l(\rho)\exp(il(\theta+\alpha)) \tag{4.10-13}$$

Then by matching the coefficients in Eq. (4.10-11) using Eqs. (4.10-9) and (4.10-10), it follows that the spectral coefficients for this case are given by

$$\left.\begin{aligned} a_l &= E_o i^l \exp(il\alpha) \\ b_l &\equiv 0 \\ c_l &= H_o i^l \sqrt{\mu/\varepsilon}\exp(il\alpha) = E_o i^l \exp(il\alpha) \\ d_l &\equiv 0 \end{aligned}\right\} \tag{4.10-14}$$

For a TM plane wave traveling in the positive z-direction in Fig. 4-9, H_y / H_o is given by Eq. (4.10-9), and E_θ / E_o and E_r / E_o are given by the forms in Eq. (4.10-10).

Using the property of the Bessel functions given by $J_{-l}(\rho) = (-1)^l J_l(\rho)$, Eq. (4.10-9) may be written as

$$\exp(i\rho\cos\theta) = J_0(\rho) + 2\sum_{l=1}^{l=\infty} i^l J_l(\rho)\cos(l\theta) \tag{4.10-9′}$$

Using the stationary phase technique, we will show in a later section that for locations well into the upper half-plane, $0 < \theta < \pi$, only spectral components involving the $\exp(-il\theta)$ part of $\cos(l\theta)$ contribute to the summations in Eq. (4.10-11) when $r/\lambda >> 1$. Moreover, these contributions come from neighborhoods around $l \approx \rho$.

For large l and ρ, we may replace the integer-order Bessel functions with their Airy function asymptotic forms (see Eqs. (3.8-1) and (3.8-2). For the TE case, the asymptotic forms for the elements of the characteristic matrix are

$$\left.\begin{aligned}
F_l(\hat{y},\hat{y}_0) &\doteq \pi\frac{K_{\rho_0}}{K_\rho}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}\\
f_l(\hat{y},\hat{y}_0) &\doteq \frac{-i\pi}{2K_\rho K_{\rho_0}}\sqrt{\frac{\mu}{\varepsilon}}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\end{vmatrix}\\
G_l(\hat{y},\hat{y}_0) &\doteq -i2\pi K_\rho K_{\rho_0}\sqrt{\frac{\varepsilon}{\mu}}\begin{vmatrix}\mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}\\
g_l(\hat{y},\hat{y}_0) &\doteq \frac{\pi K_\rho}{K_{\rho_0}}\begin{vmatrix}\mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\\ \mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\end{vmatrix}\\
\hat{y} &= K_\rho^{-1}(l-\rho)\left(1-\frac{l-\rho}{60K_\rho^3}+\cdots\right),\quad K_\rho=\left(\frac{\rho}{2}\right)^{1/3}
\end{aligned}\right\}\tag{4.10-15}$$

4.11 Wave Propagation in a Spherical Stratified Medium

Here we develop the characteristic matrix for spherical stratification, which is pursued in later sections using a modified Mie scattering theory. Spherical harmonic functions in the angular coordinates will result as solutions for the electromagnetic field when $n = n(r)$, and spherical Bessel functions also will result when the index of refraction is constant within a layer.

We consider an approaching wave that is planar with its electric vector parallel to the x-axis. Figure 4-10 shows the geometry. The central angle between the radius vector $\mathbf{r}$ and the z-axis is given by θ. The Poynting $\boldsymbol{S}$ for the approaching plane wave is directed along the z-axis. The azimuth angle ϕ lies in the yx-plane. The plane of incidence is given by $\phi = 0$.

Maxwell's curl equations for the TM wave become

$$\left.\begin{aligned}
&\frac{1}{r\sin\theta}\left(\frac{\partial E_r}{\partial\phi}-\sin\theta\frac{\partial(rE_\phi)}{\partial r}\right)=ik\mu H_\theta\\
&H_r=\left(\frac{\partial(E_\phi\sin\theta)}{\partial\theta}-\frac{\partial E_\theta}{\partial\phi}\right)\equiv 0\\
&\frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)=ik\mu H_\phi
\end{aligned}\right\}\tag{4.11-1a}$$

and

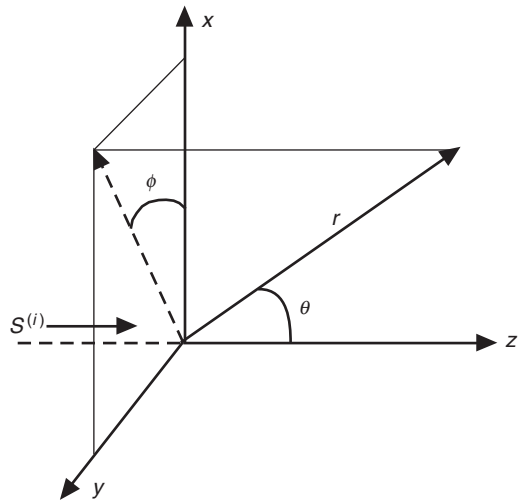

Fig. 4-10. Coordinate frame for a spherical stratified medium.

$$\left.\begin{aligned}
&\frac{1}{r\sin\theta}\left(\frac{\partial\left(H_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial H_\theta}{\partial\phi}\right)=-ik\varepsilon E_r\\
&\frac{1}{r}\left(-\frac{\partial\left(rH_\phi\right)}{\partial r}\right)=-ik\varepsilon E_\theta\\
&\frac{1}{r}\frac{\partial\left(rH_\theta\right)}{\partial r}=-ik\varepsilon E_\phi
\end{aligned}\right\}\tag{4.11-1b}$$

Since we are only concerned with in-plane propagation, it follows for the TM wave that $H_\theta = H_r = E_\phi = 0$ in the plane $\phi = 0$. In this case, Eq. (4.11-1) simplifies to

$$\frac{1}{r\sin\theta}\frac{\partial\left(H_\phi\sin\theta\right)}{\partial\theta}=-ik\varepsilon E_r \tag{4.11-2a}$$

$$\frac{1}{r}\frac{\partial\left(rH_\phi\right)}{\partial r}=ik\varepsilon E_\theta \tag{4.11-2b}$$

$$\frac{1}{r}\left(\frac{\partial\left(rE_\theta\right)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)=ik\mu H_\phi \tag{4.11-2c}$$

Using the separation of variables technique again, we see from Eq. (4.11-2b) that in the θ coordinate H_ϕ and E_θ must be represented by the same function.

But Eq. (4.11-2c) shows that this function must be equal to the derivative of the θ function used for E_r times a constant. Accordingly, we set

$$\left.\begin{aligned} H_\phi &= U(r)\Theta'(\theta) \\ E_\theta &= V(r)\Theta'(\theta) \\ E_r &= W(r)\Theta(\theta) \end{aligned}\right\} \tag{4.11-3}$$

Equations (4.11-2) and (4.11-3) require that the following coupled system of first-order differential equations be satisfied:

$$\frac{U}{r}\frac{1}{\sin\theta}\left(\frac{d(\Theta'\sin\theta)}{d\theta}\right) = -ik\varepsilon W\Theta \tag{4.11-4a}$$

$$\frac{d(rU)}{dr}\Theta' = ik\varepsilon rV\Theta' \tag{4.11-4b}$$

$$\frac{1}{r}\left(\frac{d(rV)}{dr}\Theta' - W\Theta'\right) = ik\mu U\Theta' \tag{4.11-4c}$$

We see that Eqs. (4.11-4b) and (4.11-4c) have a common factor, $\Theta'(\theta)$, which can be factored, leaving only terms involving the radial coordinate. But, for Eq. (4.11-4a) to hold for all possible values of r and θ, it is required that

$$\left.\begin{aligned} \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= \text{ constant} = -l(l+1) \\ ik\varepsilon rW(r) &= -l(l+1)U(r) \end{aligned}\right\} \tag{4.11-5}$$

The top line, of course, is the differential equation for a Legendre polynomial $P_l(\cos\theta)$. The constant in Eq. (4.11-5) has the value of $-l(l+1)$, with l restricted to integer values to ensure physical solutions to the differential equation and also single-valuedness for $\Theta(\theta)$. For negative integer values for l, we set $l = -\tilde{l} - 1$ with $\tilde{l} \geq 0$. It follows that $l(l+1) = (\tilde{l}+1)\tilde{l}$ and, therefore, $P_l = P_{-l-1}$. It follows that we need only the non-negative integers for l. The second solution to the Legendre differential equation $Q_l(\cos\theta)$ has a singularity at $\theta = 0$. Because it does not match the boundary conditions, we discard it.

From Eqs. (4.11-4) and (4.11-5), we obtain for the TM case

$$\varepsilon\frac{d}{dr}\left(\frac{1}{\varepsilon}\frac{d(rU_l)}{dr}\right) + \left(k^2n^2 - \frac{l(l+1)}{r^2}\right)(rU_l) = 0, \quad l = 0,1,2,\cdots \tag{4.11-6a}$$

$$\left.\begin{aligned}\frac{1}{kn}\frac{d(rV_l)}{dr} &= i\sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{l(l+1)}{(knr)^2}\right)(rU_l)\\ \frac{1}{kn}\frac{d(rU_l)}{dr} &= i\sqrt{\frac{\varepsilon}{\mu}}(rV_l)\end{aligned}\right\} \tag{4.11-6b}$$

$$nkrW_l = -il(l+1)\sqrt{\frac{\mu}{\varepsilon}}U_l \tag{4.11-6c}$$

$$\Theta_l = P_l(\cos\theta) \tag{4.11-6d}$$

The solution set for the TE wave follows the same course, starting with the TE version of Maxwell's curl equations that have been given in Eq. (4.11-1) for the TM case, and ending with the TE version of Eq. (4.11-6). For the TE wave, ε will be replaced by μ in the modified wave equation in Eq. (4.11-6a); the factor $-\sqrt{\varepsilon/\mu}$ replaces $\sqrt{\mu/\varepsilon}$ in the top line of Eq. (4.11-6b), and $-\sqrt{\mu/\varepsilon}$ replaces $\sqrt{\varepsilon/\mu}$ in the bottom line of Eq. (4.11-6b) and in Eq. (4.11-6c).

The superposition of the fields for the TM wave and the TE wave yields the complete field. Hence, a plane wave, invariant along the y-direction in Fig. 4-10, and with its electric field vector pointed along the x-axis and its magnetic vector along the y-axis, can be expressed in terms of the superposition of its TE and TM components. However, when $r/\lambda >> 1$ and $l >> 1$, the TE terms can be ignored for this case when the electric field vector is directed along the x-axis. This follows from inspecting the asymptotic forms for the Legendre polynomials for large l. For a TM wave, the series representation for the electric field involves $dP_l/d\theta$ terms. For large l, one can readily show from the asymptotic form for the Legendre polynomial that $dP_l/d\theta \sim lP_l$. On the other hand, for the TE wave, the series representation for the electric field involves only P_l terms, which are negligible compared to $dP_l/d\theta$. For the cylindrical representation, the TE terms are completely absent for a plane wave with its electric field vector along the x-axis.

When ε and μ are constant, the solutions to Eq. (4.11-6a) are spherical Bessel functions of the first and second kind and of integer-order l. Thus, the general solution to the system in Eq. (4.11-6) for the lth spectral component of the TM wave is given by

$$\left.\begin{aligned}&\rho U_l = \{\psi_l(\rho), \chi_l(\rho)\},\ \ \rho V_l = -i\sqrt{\frac{\mu}{\varepsilon}}\{\psi_l'(\rho), \chi_l'(\rho)\},\ \ \rho = nkr,\\ &\rho W_l = -il(l+1)\sqrt{\frac{\mu}{\varepsilon}}U_l,\ \ \Theta_l = P_l(\cos\theta),\ \ l = 0,1,2\cdots\end{aligned}\right\} \tag{4.11-7}$$

Here $\psi_l(\rho)$ is the spherical Bessel function of the first kind and of order l, and $\chi_l(x)$ is the spherical Bessel function of the second kind. Equation (3.2-5) defines $\psi_l(x)$ and $\chi_l(x)$.

The characteristic matrix for the lth spectral component is defined by

$$\rho\begin{pmatrix} U_l(\rho) \\ V_l(\rho) \end{pmatrix} = \boldsymbol{M}_l[\rho,\rho_0]\begin{pmatrix} U_l(\rho_0) \\ V_l(\rho_0) \end{pmatrix}\rho_0 \tag{4.11-8}$$

It is given for the in-plane TM case by

$$\begin{aligned} \boldsymbol{M}_l[\rho,\rho_0] &= \begin{bmatrix} F_l(\rho,\rho_0) & f_l(\rho,\rho_0) \\ G_l(\rho,\rho_0) & g_l(\rho,\rho_0) \end{bmatrix} \\ &= \begin{bmatrix} \begin{vmatrix} \psi_l(\rho) & \chi_l(\rho) \\ \psi_l'(\rho_0) & \chi_l'(\rho_0) \end{vmatrix} & -i\sqrt{\dfrac{\varepsilon}{\mu}}\begin{vmatrix} \psi_l(\rho) & \chi_l(\rho) \\ \psi_l(\rho_0) & \chi_l(\rho_0) \end{vmatrix} \\ -i\sqrt{\dfrac{\mu}{\varepsilon}}\begin{vmatrix} \psi_l'(\rho) & \chi_l'(\rho) \\ \psi_l'(\rho_0) & \chi_l'(\rho_0) \end{vmatrix} & -\begin{vmatrix} \psi_l'(\rho) & \chi_l'(\rho) \\ \psi_l(\rho_0) & \chi_l(\rho_0) \end{vmatrix} \end{bmatrix} \end{aligned} \tag{4.11-9}$$

Here the Wronskian $\psi_l\chi_l' - \psi_l'\chi_l = 1$ has been used to ensure that the determinant of $\boldsymbol{M}_l[\rho,\rho_0]$ is unity for all values of ρ. For the TE wave in the plane $\phi = 0$, i.e., the case where the electric field vector for the incident plane wave is perpendicular to the *xz*-plane in Fig. 4-7 ($H_y \equiv 0$), the characteristic matrix is the same as that given in Eq. (4.11-9) except that the reciprocals of $\sqrt{\varepsilon/\mu}$ and $\sqrt{\mu/\varepsilon}$ are used in the off-diagonal elements.

For large l and ρ, we may replace the spherical Bessel functions with their Airy function asymptotic forms [see Eqs. (3.8-1) and (3.8-2)]. The asymptotic forms for the elements of the characteristic matrix for the case of the electric field vector of the incident planar wave directed along the *x*-axis in Fig. 4-10 are

$$\left.\begin{aligned}
F_l(\hat{y},\hat{y}_0) &\doteq \pi\sqrt{\frac{K_\rho}{K_{\rho_o}}}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}\\
f_l(\hat{y},\hat{y}_0) &\doteq i\pi\sqrt{K_\rho K_{\rho_o}}\sqrt{\frac{\varepsilon}{\mu}}\begin{vmatrix}\mathrm{Ai}[\hat{y}] & \mathrm{Bi}[\hat{y}]\\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\end{vmatrix}\\
G_l(\hat{y},\hat{y}_0) &\doteq i\frac{\pi}{\sqrt{K_\rho K_{\rho_o}}}\sqrt{\frac{\mu}{\varepsilon}}\begin{vmatrix}\mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\\ \mathrm{Ai}'[\hat{y}_0] & \mathrm{Bi}'[\hat{y}_0]\end{vmatrix}\\
g_l(\hat{y},\hat{y}_0) &\doteq -\pi\sqrt{\frac{K_{\rho_o}}{K_\rho}}\begin{vmatrix}\mathrm{Ai}'[\hat{y}] & \mathrm{Bi}'[\hat{y}]\\ \mathrm{Ai}[\hat{y}_0] & \mathrm{Bi}[\hat{y}_0]\end{vmatrix}
\end{aligned}\right\} \tag{4.11-10}$$

where

$$\left.\begin{aligned}
\hat{y} &= \nu^{2/3}\zeta[\nu/\rho] \quad \text{(see Section 3.8)}\\
&= K_\rho^{-1}(\nu-\rho)(1-K_\rho^{-3}(\nu-\rho)/60+\cdots\\
K_\rho &= (\rho/2)^{1/3}; \quad \nu = l+1/2
\end{aligned}\right\} \tag{4.11-11}$$

Given specified beginning and end points (r, r_0), one would perform a weighted integration of each of the matrix elements in Eq. (4.11-10) with respect to $\hat{y}$ to obtain the complete asymptotic form for the characteristic matrix. The weights might be the spectral coefficients of the wave at r_0. For example, for an approaching plane wave at (r_0, θ_0) along the z-axis, one has

$$\left.\begin{aligned}
\rho_0 U_l(\rho_0) &= H_0\, i^l \frac{2l+1}{l(l+1)}\psi_l(\rho_0)\\
\rho_0 V_l(\rho_0) &= E_0\, i^{l-1}\frac{2l+1}{l(l+1)}\psi_l'(\rho_0)\\
\rho_0^2 W_l(\rho_0) &= E_0\, i^{l-1}(2l+1)\psi_l(\rho_0)
\end{aligned}\right\} \tag{4.11-12}$$

Substituting these forms into Eq. (4.11-8) for $U_l(\rho_0)$ and $V_l(\rho_0)$ leads, of course, to the identical forms for $U_l(\rho)$ and $V_l(\rho)$ with ρ_0 replaced by ρ. Since we started with a planar form for the approaching wave, it must remain planar when n is held constant within the medium.

The characteristic matrix in Eq. (4.11-10) for the spherical case is essentially the same as the TM version for the cylindrical case given in Eq. (4.11-8), except for the scale factors K_ρ.

4.12 Correspondence between Characteristic Matrices for Cartesian and Spherical Stratified Airy Layers

The elements of the characteristic matrix in Eq. (4.11-10) are the asymptotic forms for large values of l and ρ; they provide a two-dimensional description in polar coordinates of the propagation of an in-plane polarized wave ($E_y \equiv 0$) through a medium of constant index of refraction. Similarly, the elements given in Eq. (4.5-14) describe a TM wave ($E_y \equiv 0$) propagating through a Cartesian stratified Airy layer. We can establish a duality between these two systems. If we assume in the spherical framework that the difference between the beginning and end points of the ray is small compared to the radius of curvature r_0 of the stratified surface, that is, $(r - r_0)/r_0 << 1$, then the same mathematics applies to both systems; a correspondence between parameters is given by

$$\left.\begin{array}{ccc} \gamma = (2k^{-1}nn')^{1/3} & \leftrightarrow & nK_\rho^{-1} = n(2/\rho)^{1/3} \\ \text{Cartesian} \uparrow \text{Stratified} & & \text{Spherical} \uparrow \text{Stratified} \\ \text{Airy Layer} & & n = \text{constant within layer; } \rho,\ l >> 1 \end{array}\right\} \quad (4.12\text{-}1)$$

The transformation between the arguments of the Airy functions in the two systems, the $\hat{y}$ values, is given by

$$\left.\begin{array}{ccc} -k\int_{x_o}^{x} \gamma dx = \hat{y} & \leftrightarrow & \hat{y} \doteq \left\{\begin{array}{l} K_\rho^{-1}(\nu - \rho) \\ \text{or} \\ K_\nu^{-4}\left(\nu^2 - \rho^2\right)/4 \end{array}\right\} \\ \text{Cartesian} \uparrow \text{Airy} & & \text{Large} \uparrow \text{Spherical} \\ \text{Layer (CAL)} & & \text{Layer (LSL)} \end{array}\right\} \quad (4.12\text{-}2a)$$

where γ is given by Eqs. (4.5-3) and (4.5-4). When $\nu \approx \rho >> 1$, $\hat{y}$ in Eq. (4.12-2a) may be expressed as

$$\left.\begin{array}{ccc} \hat{y} = -\gamma^{-2}\left(n^2 - n_o^2\right) & \leftrightarrow & \hat{y} \doteq -K_\nu^{-4}\left(\rho^2 - \nu^2\right)/4 \\ \hat{y} = -\gamma^{-2}n^2\cos^2\varphi & \leftrightarrow & \hat{y} \doteq -K_\nu^2\cos^2\theta_\nu \\ \text{CAL} \uparrow & & \uparrow \text{LSL} \end{array}\right\} \quad (4.12\text{-}2b)$$

Here we have set the spectral number $\nu = \rho\sin\theta_\nu$. Thus, θ_ν, which is the analog of φ in the Cartesian framework, may be interpreted as the "angle of

incidence" of the νth spectral component in the spherical framework. Chapters 3 and 5 discuss θ_ν (see Fig. 3-14) in a wave theory context. These chapters also address the role of θ_ν in the connections between wave theory and geometric optics, and in stationary phase theory.

In Eq. (4.12-1), we can write γ for an Airy layer in the form

$$\gamma = \left|\frac{2nn'}{k}\right|^{1/3} = n\left(\frac{2}{\rho}\right)^{1/3}\left|\frac{n'r}{n}\right|^{1/3} = \frac{n\beta^{1/3}}{K_\rho} \tag{4.12-3}$$

In a geometric optics framework with spherical stratification, $|n'r/n|$ is the ratio of the radius of curvature of the stratified surface r to the radius of curvature of the refracted ray $|n/n'|$. In Chapter 2, it is defined as $(-\beta)$. For the Earth's atmosphere at sea level, this ratio has a value of about 1/4 for the dry air component. For water vapor layers, it can exceed unity.

This correspondence between the forms in Eqs. (4.12-1) and (4.12-2) allows us to perform a "flattening" transformation. We may convert a series of large concentric spherical layers (with the refractivity constant within a layer but discontinuous across boundaries) into an equivalent Cartesian stratified medium consisting of a stack of Airy layers.

A similar correspondence applies when one has an Airy layer with spherical stratification. Here we have a linearly varying index of refraction, $n = n_0 + n'(r - r_0)$, where n_0 and n' are constants within a layer. Referring to the modified Bessel equation in Eq. (4.10-3), if we expand ρ and l about $\rho_0 = kn_0r_0$ while retaining only first-order terms, we obtain

$$\left.\begin{aligned} \frac{d^2U_l}{d\rho^2} &\doteq \frac{2}{\rho_0}\Big((l-\rho_0)-(\rho-\rho_0)+\beta(\rho-\rho_0)\Big)U_l \\ \beta &= -n'r_0/n_0 \end{aligned}\right\} \tag{4.12-4}$$

Here U_l is a modified Bessel function of order l, and $\beta = -n'r_0/n_0$ is a constant, which, as mentioned earlier, will be substantially smaller than unity when the thin-atmosphere assumption applies. Also, we assume that $n'(r - r_0)$ is sufficiently small so that the quadratic term for $n(r)$ can be ignored in Eq. (4.12-4). If we now make the transformation

$$\tilde{y} = \frac{(2/\rho_0)^{1/3}}{|1-\beta|^{2/3}}\big(l - \rho + \beta(\rho - \rho_0)\big) \tag{4.12-5}$$

it follows that Eq. (4.12-4) becomes

$$\frac{d^2U}{d\tilde{y}^2} - \tilde{y}U = 0 \tag{4.12-6}$$

which is the differential equation for the Airy functions. It follows that the correspondence in Eq. (4.12-2) is modified for the spherical stratified Airy case. In particular, the parameters $K_\nu^{-1} \doteq \partial\tilde{y} / \partial\nu$ and $K_\rho^{-1} \doteq \partial\tilde{y} / \partial\rho$ for the case of constant refractivity are modified for an Airy layer to

$$\left.\begin{aligned} \frac{\partial\tilde{y}}{\partial\rho} &\doteq -K_{\rho_o}^{-1}(1-\beta)^{1/3} \\ \frac{\partial\tilde{y}}{\partial\nu} &\doteq K_{\rho_o}^{-1}(1-\beta)^{-2/3} \end{aligned}\right\} \tag{4.12-7}$$

Notice that this transformation in Eq. (4.12-5) for $\tilde{y}$ also works for $\beta > 1$, a region of super-refractivity. In this case, we have

$$\left.\begin{aligned} \tilde{y} &= \frac{1}{K_{\rho_o}(\beta-1)^{2/3}}\left(l - \rho + \beta(\rho - \rho_0)\right) \\ \frac{\partial\tilde{y}}{\partial\rho} &\doteq K_{\rho_o}^{-1}(\beta-1)^{1/3} > 0 \end{aligned}\right\} \tag{4.12-8}$$

In summary, for spherical or cylindrical stratification with $r_0 / \lambda >> 1$, a stack of concentric layers with the refractivity held constant within a layer, but allowed to vary from layer to layer, leads to the same mathematical expressions for the lth spectral component of wave propagation as those expressions obtained from a stack of Airy layers with Cartesian stratification with an angle of incidence determined from Eq.(4.12-2b). Moreover, to the extent that a stack of Airy layers can be made through the first-order approximation of the characteristic matrix to closely replicate wave propagation through a Cartesian stratified medium with an arbitrarily varying refractivity profile, this duality should yield similar accuracy for the spherical or cylindrical stratification, also with an arbitrarily varying refractivity profile. Thus, the problem of wave propagation in large cylindrical or spherical stratified media can be transformed to an analogous problem of wave propagation in a Cartesian stratified medium. Also, summing over l in cylindrical or spherical stratification is equivalent to integrating over n_o in an Airy layer in the Cartesian stratified medium. Integrating over n_o is equivalent to integrating over angle of incidence or "impact parameter."

For a given spectral number in the curvilinear system, the relationship between angles of incidence in the two systems is given by

$$v = \rho \sin\theta_v, \quad \cos\theta_v = /\, n' r \,/\, n \,/^{-1/3} \cos\varphi = \beta^{-1/3} \cos\varphi \qquad (4.12\text{-}9)$$

If we force equality in angles of incidence in the two systems, then Eq. (4.12-9) yields a value for n' to be used in the surrogate Cartesian stratified model using Airy layers. This value is $n' = n / r$, the magnitude of the super-refractivity critical gradient. In this case, it follows that γ for the Cartesian Airy layer is given by $\gamma = n_o / K_{\rho_o}$.

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