Appendix I
Field Equations in a Stratified Medium

The electromagnetic field vectors for the homogeneous medium are obtained from the vector calculus operations on the scalar potentials given in Eq. (5.2-7). To show that these expressions actually yield a vector field that satisfies Maxwell's equations, we take the curl and divergence of these forms. For example, for the transverse electric (TE) case, we have $\boldsymbol{E}_{\mathrm{TE}} = ik\mu\nabla\times[{}^{\mathrm{m}}\Pi\boldsymbol{r}]$ and $\boldsymbol{H}_{\mathrm{TE}} = \nabla\times\nabla\times[{}^{\mathrm{m}}\Pi\boldsymbol{r}]$. Taking the curl of $\boldsymbol{E}_{\mathrm{TE}}$ evidently satisfies Eq. (5.2-2a). Also, $\nabla\cdot\boldsymbol{H}_{\mathrm{TE}} \equiv 0$, which satisfies Eq. (5.2-2d). Also, $\nabla\cdot\boldsymbol{E}_{\mathrm{TE}} \equiv 0$; therefore, Eq. (5.2-2c) is satisfied. Finally, taking the curl of $\boldsymbol{H}_{\mathrm{TE}}$, $\nabla\times\boldsymbol{H}_{\mathrm{TE}} = \nabla\times(\nabla\times(\nabla\times[{}^{\mathrm{m}}\Pi\boldsymbol{r}]))$, one may use the identities $\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ and $\nabla^2(\nabla\times(\psi\boldsymbol{r})) = \nabla\times(\nabla^2(\psi\boldsymbol{r})) = -\boldsymbol{r}\times\nabla(\nabla^2\psi)$ and apply the Helmholtz equation in Eq. (5.2-6) to obtain Eq. (5.2-2b). By symmetry arguments, Maxwell's equations also hold for the transverse magnetic (TM) case where $\boldsymbol{E}_{\mathrm{TM}} = \nabla\times\nabla\times[{}^{\mathrm{e}}\Pi\boldsymbol{r}]$ and $\boldsymbol{H}_{\mathrm{TM}} = -ik\varepsilon\nabla\times[{}^{\mathrm{e}}\Pi\boldsymbol{r}]$.

For the stratified medium with $n = n(r)$, the electromagnetic vector field can be derived from scalar potentials that are solutions to a modified Helmholtz equation. As discussed in Section 5.2, the scalar potentials for the stratified medium have some degree of freedom in their definition. The electromagnetic field vectors for the case where $n = n(r)$ can be expressed through vector calculus operations on a vector version for the modified scalar potentials, $\left[{}^{\mathrm{e}}\Pi\varepsilon^{1/2}\boldsymbol{r}\right]$ and $\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right]$. The field vectors are given by

$$\left.\begin{aligned}\boldsymbol{E} &= \varepsilon^{-1}\nabla\times\nabla\times\left[{}^{\mathrm{e}}\Pi\varepsilon^{1/2}\boldsymbol{r}\right] + ik\nabla\times\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right]\\ \boldsymbol{H} &= \mu^{-1}\nabla\times\nabla\times\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right] - ik\nabla\times\left[{}^{\mathrm{e}}\Pi\varepsilon^{1/2}\boldsymbol{r}\right]\end{aligned}\right\}\tag{I-1}$$

The factors $\varepsilon^{1/2}$ and $\mu^{1/2}$ have been inserted into the potential terms in Eq. (I-1) to simplify the resulting modified Helmholtz equation that each of the scalar potentials must satisfy. These are given by

$$\left.\begin{aligned}\nabla^2({}^{\mathrm{e}}\Pi) + k^2\tilde{n}_{\mathrm{TM}}^2\,{}^{\mathrm{e}}\Pi = 0\\ \nabla^2({}^{\mathrm{m}}\Pi) + k^2\tilde{n}_{\mathrm{TE}}^2\,{}^{\mathrm{m}}\Pi = 0\end{aligned}\right\} \tag{I-2}$$

where the modified indices of refraction are given by

$$\left.\begin{aligned}\tilde{n}_{\mathrm{TM}}^2 = n^2\left[1 - \frac{r^2\varepsilon^{1/2}}{k^2n^2}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{\varepsilon^{1/2}}\right)\right)\right]\\ \tilde{n}_{\mathrm{TE}}^2 = n^2\left[1 - \frac{r^2\mu^{1/2}}{k^2n^2}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{\mu^{1/2}}\right)\right)\right]\end{aligned}\right\} \tag{I-3}$$

In the electrodynamics literature, the TE and TM waves are generated from linearly independent solutions to the Helmholtz equation. In Eq. (I-1), the term $ik\nabla\times\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right]$ generates the electric field $\boldsymbol{E}_{\mathrm{TE}}$, which is perpendicular to $\boldsymbol{r}$, that is, a transverse electric field; this wave is known in the literature as the TE wave. Similarly, the term $-ik\nabla\times\left[{}^{\mathrm{e}}\Pi\varepsilon^{1/2}\boldsymbol{r}\right]$ generates a transverse magnetic field $\boldsymbol{H}_{\mathrm{TM}}$, or the TM wave. The converse polarization holds for the field vectors generated from the curl operations on the $\left[{}^{\mathrm{e}}\Pi\varepsilon^{1/2}\boldsymbol{r}\right]$ term.

If one forms the curl and divergence of $\boldsymbol{E}$ and $\boldsymbol{H}$ as defined by Eq. (I-1), one can show that these scalar potentials do indeed generate fields that satisfy Maxwell's equations in Eq. (5.2-2) when $\varepsilon = \varepsilon(r)$ and $\mu = \mu(r)$. For the TE case, we have $\boldsymbol{E}_{\mathrm{TE}} = ik\nabla\times\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right]$ and $\boldsymbol{H}_{\mathrm{TE}} = \mu^{-1}\nabla\times\nabla\times\left[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}\right]$. Taking the curl of $\boldsymbol{E}_{\mathrm{TE}}$ evidently satisfies Eq. (5.2-2a). Also, $\nabla\cdot\left(\mu\boldsymbol{H}_{\mathrm{TE}}\right)\equiv 0$, which satisfies Eq. (5.2-2d). Noting that $\nabla\cdot\boldsymbol{E}_{\mathrm{TE}}\equiv 0$, we have $\nabla\cdot\left(\varepsilon\boldsymbol{E}_{\mathrm{TE}}\right) = \nabla\varepsilon\cdot\boldsymbol{E}_{\mathrm{TE}}$, which is zero because $\nabla\varepsilon$ is radial directed; therefore, Eq. (5.2-2c) is satisfied. Finally, for the case where $\nabla\times\boldsymbol{H}_{\mathrm{TE}} = \nabla\times(\mu^{-1}\nabla\times\nabla\times[{}^{\mathrm{m}}\Pi\mu^{1/2}\boldsymbol{r}])$, one may use the identities $\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ and $\nabla^2(\nabla\times(\psi\boldsymbol{r})) = \nabla\times(\nabla^2(\psi\boldsymbol{r})) = -\boldsymbol{r}\times\nabla(\nabla^2\psi)$ and apply Eq. (I-2) to obtain Eq. (5.2-2b). By symmetry, the same proof applies to the TM case.