APPENDIX A

LOCATING THE LAGRANGE POINTS

A.1 INTRODUCTION

The discussion given here, previously authored by Parker [46], is devoted to deriving analytical expressions for the Lagrange points in the circular restricted three-body problem (CRTBP). Szebehely provides more details and a clear description of this derivation [86]. Other authors have provided similar derivations, including Moulton [106] and Broucke et al. [236].

A.2 SETTING UP THE SYSTEM

Let us begin with a system of two masses, m_1 and m_2, such that $m_1 \geq m_2$. Furthermore, each of these masses is orbiting the center of mass of the system in a circle. Then there exist cases where a third body, m_3, of negligible mass can be placed in the system in such a way that the force of gravity from both bodies and the rotational motion in the system balance to produce a configuration that does not change in time with respect to the rotating system. That is, each body rotates about the center of mass at exactly the same rate and is seemingly fixed in the rotating frame of reference. Euler and Lagrange located five of these cases, and those locations have henceforth been known as the five Lagrange points in a three-body system.

To locate the Lagrange points, we begin with the three bodies stationary in the corotating frame of reference. That is

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}(t) \tag{A.1}$$

where $\dot{\theta}_i$ is the angular velocity of the body of mass m_i about the center of mass. Furthermore, if the shape of the configuration does not alter over time, the relative distances $r_{12}(t)$, $r_{23}(t)$, and $r_{31}(t)$ are given by

$$\frac{r_{12}(t)}{r_{12}(t_0)} = \frac{r_{23}(t)}{r_{23}(t_0)} = \frac{r_{31}(t)}{r_{31}(t_0)} = f(t) \tag{A.2}$$

So far, there are no constraints on the relative size of the configuration, only on the angular velocity and the shape of the configuration.

Next, we move the origin to the center of mass of the configuration. Then $\vec{R}_i$ describes the vector position of the i^{th} mass, satisfying the constraint

$$\sum_{i=1}^{3} m_i \vec{R}_i = 0 \tag{A.3}$$

Equation (A.3) may be written

$$(m_1 + m_2 + m_3)\vec{R}_1 + m_2(\vec{R}_2 - \vec{R}_1) + m_3(\vec{R}_3 - \vec{R}_1) = 0,$$

or

$$M\vec{R}_1 = -m_2\vec{r}_{12} - m_3\vec{r}_{13} \tag{A.4}$$

where M is equal to the sum of the masses in the system. Squaring this relationship produces

$$M^2 R_1^2 = m_2^2 r_{12}^2 + m_3^2 r_{13}^2 + 2m_2 m_3 \vec{r}_{12} \bullet \vec{r}_{13} \tag{A.5}$$

where R_i and r_i denote the magnitudes of the vectors $\vec{R}_i$ and $\vec{r}_i$, respectively. Since we know that the relative shape of the configuration does not change, as seen above, we may substitute in the relationships for the relative angles and distances (Eqs. (A.1) and (A.2)) into Eq. (A.5) to find that, in general

$$R_i(t) = R_i(t_0) f(t) \tag{A.6}$$

If F_i is the magnitude of the force per unit mass acting on the mass m_i, then the total force acting on m_i is $m_i F_i$ and the equation of motion of that mass along the direction of the force satisfies

$$m_i F_i = m_i \left(\ddot{R}_i - R_i \dot{\theta}_i^2\right) \tag{A.7}$$

Since all of the particles are rotating at the same rate, we can reduce this relationship to the following

$$m_i F_i = m_i \quad R_i(t_0)\ddot{f}(t) - R_i \dot{\theta}^2 \Big]$$

or equivalently

$$m_i F_i = R_i m_i \quad \ddot{f}(t)/f(t) - \dot{\theta}^2 \Big] \tag{A.8}$$

Hence, we have the proportionality relationship

$$F_1 : F_2 : F_3 = R_1 : R_2 : R_3 \tag{A.9}$$

There are two cases that will satisfy the conditions given in Eqs. (A.8) and (A.9). The two cases are

$$\vec{R}_i \times \vec{F}_i = 0 \qquad \text{or} \qquad \vec{R}_i \times \ddot{\vec{R}}_i = 0 \tag{A.10}$$

When we set $i = 1$ and look at the first particle, we have the following force function

$$m_1 \ddot{\vec{R}}_1 = G \quad \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12} + \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} \tag{A.11}$$

When we take the cross product of $\vec{R}_1$ with each side of Eq. (A.11), we obtain the following expression

$$\vec{R}_1 \times \quad \frac{m_2}{r_{12}^3} \vec{R}_2 + \frac{m_3}{r_{13}^3} \vec{R}_3 \quad = 0 \tag{A.12}$$

Using the center of mass relationship given in Eq. (A.3), this can be simplified to

$$m_2 \vec{R}_1 \times \vec{R}_2 \quad \frac{1}{r_{12}^3} - \frac{1}{r_{13}^3} \quad = 0 \tag{A.13}$$

Once, again, there are two similar equations for the other two particles. For Eq. (A.13) to hold, either of the following expressions must be true

$$r_{12} = r_{23} = r_{31} = r \tag{A.14}$$

(the equilateral triangle solution), or

$$\vec{R}_1 \times \vec{R}_2 = \vec{R}_2 \times \vec{R}_3 = \vec{R}_3 \times \vec{R}_1 = 0 \tag{A.15}$$

(the collinear solution).

The triangular and collinear cases are addressed separately in Sections A.3 and A.4.

A.3 TRIANGULAR POINTS

In the equilateral triangle case given in Eq. (A.14), we arrive at the following relationship for the first particle

$$\ddot{\vec{R}}_1 + G M_1 \frac{\vec{R}_1}{R_1^3} = 0 \tag{A.16}$$

where

$$M_1 = \frac{m_2^2 + m_3^2 + m_2 m_3 \;\; ^{3/2}}{(m_1 + m_2 + m_3)^2} \tag{A.17}$$

This result is the familiar two-body equation of motion. In this case, the first particle moves about the center of mass of the system in any conic orbit as if it had unit mass and a mass of M_1 were placed at the center of mass of the system. Each particle moves in a corresponding trajectory, and the figure remains in an equilateral triangle configuration (although its size may oscillate or grow indefinitely).

A.4 COLLINEAR POINTS

In the collinear case given in Eq. (A.15), we can also first show that each particle's orbit is a conic section. Beginning with the first particle, we can take the collinear axis to be the x axis; the force acting on m_1 is then

$$F_1 = m_2 \frac{(x_2 - x_1)}{x_{12}^3} + m_3 \frac{(x_3 - x_1)}{x_{13}^3} \tag{A.18}$$

But we also know from Eq. (A.6) that

$$x_i(t) = x_i(t_0) f(t)$$

so that

$$F_1 = \frac{1}{f^2} \quad m_2 \frac{(x_2 - x_1)}{x_{12}^3} + m_3 \frac{(x_3 - x_1)}{x_{13}^3} \quad _0 = \frac{\text{constant}}{f^2} \tag{A.19}$$

Since f is proportional to distance, m_1 is acted upon by an inverse-square-law central force. Hence, the particle's orbit is a conic section.

Now we will impose the condition from Eq. (A.9) that

$$F_1 : F_2 : F_3 = x_1 : x_2 : x_3.$$

This condition introduces the proportionality constant A, such that

$$\begin{aligned} F_1 &= A x_1 \\ F_2 &= A x_2 \\ F_3 &= A x_3 \end{aligned} \tag{A.20}$$

or equally

$$\begin{aligned} A x_1 &= m_2 \frac{x_2 - x_1}{x_{12}^3} + m_3 \frac{x_3 - x_1}{x_{13}^3} \\ A x_2 &= m_3 \frac{x_3 - x_2}{x_{23}^3} + m_1 \frac{x_1 - x_2}{x_{21}^3} \\ A x_3 &= m_1 \frac{x_1 - x_3}{x_{31}^3} + m_2 \frac{x_2 - x_3}{x_{32}^3} \end{aligned} \tag{A.21}$$

We are looking for the placement of the particle of mass m_3 with respect to the other two particles such that the relative positions are constant in the rotating frame. The equilibrium positions possible for m_3 are in the arrangements $m_1 - m_3 - m_2$ (case 132), $m_1 - m_2 - m_3$ (case 123), and $m_3 - m_1 - m_2$ (case 312). Each case will be observed separately.

A.4.1 Case 132: Identifying the L_1 point

For case 132, we are looking for a positive value of X such that

$$\left.\begin{aligned} X &= \frac{x_2 - x_3}{x_3 - x_1} = \frac{x_{32}}{x_{13}} \\ X + 1 &= \frac{x_2 - x_1}{x_3 - x_1} = \frac{x_{12}}{x_{13}} \end{aligned}\right\} \tag{A.22}$$

We identify X using a series of steps. We first subtract Ax_1 from Ax_3 and Ax_3 from Ax_2 from Eq. (A.21) to arrive at Ax_{13} and Ax_{32}

$$\begin{aligned} Ax_{13} &= -\frac{m_1 + m_3}{x_{13}^2} + m_2 \quad \frac{1}{x_{32}^2} - \frac{1}{x_{12}^2} \\ Ax_{32} &= -\frac{m_2 + m_3}{x_{32}^2} + m_1 \quad \frac{1}{x_{13}^2} - \frac{1}{x_{12}^2} \end{aligned} \tag{A.23}$$

Using Eq. (A.22), we know that $x_{32} = Xx_{13}$ and $x_{12} = (X + 1)x_{13}$. When we substitute these relationships into Eq. (A.23), we find two different relationships for the quantity Ax_{13}^3. When we set them equal and arrange in powers of X, we arrive at Lagrange's quintic equation

$$\begin{aligned} &(m_1 + m_3)X^5 + (3m_1 + 2m_3)X^4 + (3m_1 + m_3)X^3 \\ &\qquad - (3m_2 + m_3)X^2 - (3m_2 + 2m_3)X - (m_2 + m_3) = 0 \end{aligned} \tag{A.24}$$

We can use a quintic solver to solve for X (see Section A.5). Since the coefficients of Eq. (A.24) change sign only once, there can be only one positive real root. We can then use that value for X to determine the relative location of the massless particle, that is, the location of L_1, with respect to the other two particles by solving for x_3 in Eq. (A.22)

$$X = \frac{x_2 - x_3}{x_3 - x_1} \qquad \Rightarrow \qquad x_3 = x_1 + \frac{x_2 - x_1}{X + 1} \tag{A.25}$$

A.4.2 Case 123: Identifying the L_2 point

For case 123, we are looking for a positive value of X such that

$$\left.\begin{aligned} X &= \frac{x_3 - x_2}{x_2 - x_1} = \frac{x_{23}}{x_{12}} \\ X + 1 &= \frac{x_3 - x_1}{x_2 - x_1} = \frac{x_{13}}{x_{12}} \end{aligned}\right\} \tag{A.26}$$

In order to identify X, we follow a similar derivation as in case 132. We first subtract Ax_2 from Ax_3 and Ax_1 from Ax_2 from Eq. (A.21) to arrive at Ax_{23} and Ax_{12}

$$\begin{aligned} Ax_{23} &= -\frac{m_2 + m_3}{x_{23}^2} + m_1 \quad \frac{1}{x_{12}^2} - \frac{1}{x_{13}^2} \\ Ax_{12} &= -\frac{m_1 + m_2}{x_{12}^2} + m_3 \quad \frac{1}{x_{23}^2} - \frac{1}{x_{13}^2} \end{aligned} \tag{A.27}$$

We then substitute in X and $(X+1)$ from Eq. (A.26) as before, eliminate Ax_{12}^3 between the resulting equations and arrange in powers of X to produce Lagrange's quintic equation

$$\begin{aligned} &(m_1 + m_2)X^5 + (3m_1 + 2m_2)X^4 + (3m_1 + m_2)X^3 \\ &\qquad - (m_2 + 3m_3)X^2 - (2m_2 + 3m_3)X - (m_2 + m_3) = 0 \end{aligned} \tag{A.28}$$

Once again, we can use a quintic solver to solve for X (see Section A.5), knowing that again there is only one real positive root. We can then use that value for X to determine the relative location of the massless particle, that is, the location of L_2, with respect to the other two particles by solving for x_3 in Eq. (A.26)

$$X = \frac{x_3 - x_2}{x_2 - x_1} \qquad \Rightarrow \qquad x_3 = x_2 + X(x_2 - x_1) \tag{A.29}$$

A.4.3 Case 312: Identifying the L_3 point

For case 312, we are looking for a positive value of X such that

$$\left.\begin{aligned} X &= \frac{x_2 - x_1}{x_1 - x_3} = \frac{x_{12}}{x_{31}} \\ X + 1 &= \frac{x_2 - x_3}{x_1 - x_3} = \frac{x_{32}}{x_{31}} \end{aligned}\right\} \tag{A.30}$$

In order to identify X, we follow a similar derivation as in case 132. We first subtract Ax_1 from Ax_2 and Ax_3 from Ax_1 from Eq. (A.21) to arrive at Ax_{12} and Ax_{31}

$$\begin{aligned} Ax_{31} &= -\frac{m_1 + m_3}{x_{31}^2} + m_2 \quad \frac{1}{x_{12}^2} - \frac{1}{x_{32}^2} \\ Ax_{12} &= -\frac{m_1 + m_2}{x_{12}^2} + m_3 \quad \frac{1}{x_{31}^2} - \frac{1}{x_{32}^2} \end{aligned} \tag{A.31}$$

We then substitute in X and $(X+1)$ from Eq. (A.30) as before, eliminate Ax_{31}^3 between the resulting equations and arrange in powers of X to produce Lagrange's quintic equation

$$\begin{aligned} &(m_1 + m_3)X^5 + (2m_1 + 3m_3)X^4 + (m_1 + 3m_3)X^3 \\ &\qquad - (m_1 + 3m_2)X^2 - (2m_1 + 3m_2)X - (m_1 + m_2) = 0 \end{aligned} \tag{A.32}$$

Once again, we can use a quintic solver to solve for X (see Section A.5), knowing that again there is only one real positive root. We can then use that value for X to determine the relative location of the massless particle, that is, the location of L_3, with respect to the other two particles by solving for x_3 in Eq. (A.30)

$$X = \frac{x_2 - x_1}{x_1 - x_3} \qquad \Rightarrow \qquad x_3 = x_1 - \frac{x_2 - x_1}{X} \tag{A.33}$$

A.5 ALGORITHMS

The quintics given in Eqs. (A.25), (A.29), and (A.33) provide analytic determinations of the locations of the first, second, and third Lagrange points, respectively, in the circular restricted three-body system. Szebehely outlines a fixed-point iterative scheme that may be implemented to identify the single positive real root of each of the quintic equations [86]. The fourth and fifth Lagrange points make equilateral triangles with the primaries; hence, their locations are easily determined using geometry.

Sections A.5.1–A.5.3 provide pseudo-code that may be used to implement a fixed-point iterative scheme to find the x-coordinate of L_1 – L_3, respectively. The coordinate axis and the definition of μ are defined in Section 2.5.1.

A.5.1 Numerical Determination of L_1

$$\gamma_0 = \left(\frac{\mu(1-\mu)}{3} \right)^{1/3}$$

$$\gamma = \gamma_0 + 1$$

while$|\gamma - \gamma_0| > $ tol

$$\qquad \gamma_0 = \gamma$$

$$\qquad \gamma = \left(\frac{\mu(\gamma_0 - 1)^2}{3 - 2\mu - \gamma_0(3 - \mu - \gamma_0)} \right)^{1/3}$$

endwhile

$$x_{L_1} = 1 - \mu - \gamma$$

A.5.2 Numerical Determination of L_2

$$\gamma_0 = \quad \frac{\mu(1-\mu)}{3} \quad {}^{1/3}$$

$$\gamma = \gamma_0 + 1$$

$$\text{while}|\gamma - \gamma_0| > \text{tol}$$

$$\qquad \gamma_0 = \gamma$$

$$\qquad \gamma = \quad \frac{\mu(\gamma_0 + 1)^2}{3 - 2\mu + \gamma_0(3 - \mu + \gamma_0)} \quad {}^{1/3}$$

$$\text{endwhile}$$

$$x_{L_2} = 1 - \mu + \gamma$$

A.5.3 Numerical Determination of L_3

$$\gamma_0 = \quad \frac{\mu(1-\mu)}{3} \quad {}^{1/3}$$

$$\gamma = \gamma_0 + 1$$

$$\text{while}|\gamma - \gamma_0| > \text{tol}$$

$$\qquad \gamma_0 = \gamma$$

$$\qquad \gamma = \quad \frac{(1-\mu)(\gamma_0 + 1)^2}{1 + 2\mu + \gamma_0(2 + \mu + \gamma_0)} \quad {}^{1/3}$$

$$\text{endwhile}$$

$$x_{L_3} = -\mu - \gamma$$