Appendix A
Tilted Small Perturbation Model Details

Consider a tilted surface element as shown in Fig. A-1. The plane of incidence is the (x,z) plane. The coordinate system in which the radar measures the scattering coefficients is the (h,v,k) system and is described by

$$\begin{aligned}
\hat{\mathbf{k}} &= -\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{z}} \\
\hat{\mathbf{h}} &= \frac{-\hat{\mathbf{k}}\times\hat{\mathbf{z}}}{\left|\hat{\mathbf{k}}\times\hat{\mathbf{z}}\right|} = -\hat{\mathbf{y}} \\
\hat{\mathbf{v}} &= \frac{-\hat{\mathbf{h}}\times\hat{\mathbf{k}}}{\left|\hat{\mathbf{h}}\times\hat{\mathbf{k}}\right|} = -\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}}
\end{aligned} \quad . \tag{A.1}$$

We shall refer to the radar coordinate system as the global coordinate system and the tilted coordinate system as the local coordinate system. The local and global incidence angles are defined as

$$\cos\theta_l = -\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}_l \,. \tag{A.2}$$

and

$$\cos\theta = -\hat{\mathbf{k}}\cdot\hat{\mathbf{z}} \,. \tag{A.3}$$

We define the local coordinate system such that the local z-axis coincides with the local normal vector of the tilted surface. Let α denote the tilt angle in the range direction; that is, in the $x-z$ plane. We define positive values of α as those that tilt the surface towards the radar. Similarly, let β denote the tilt in

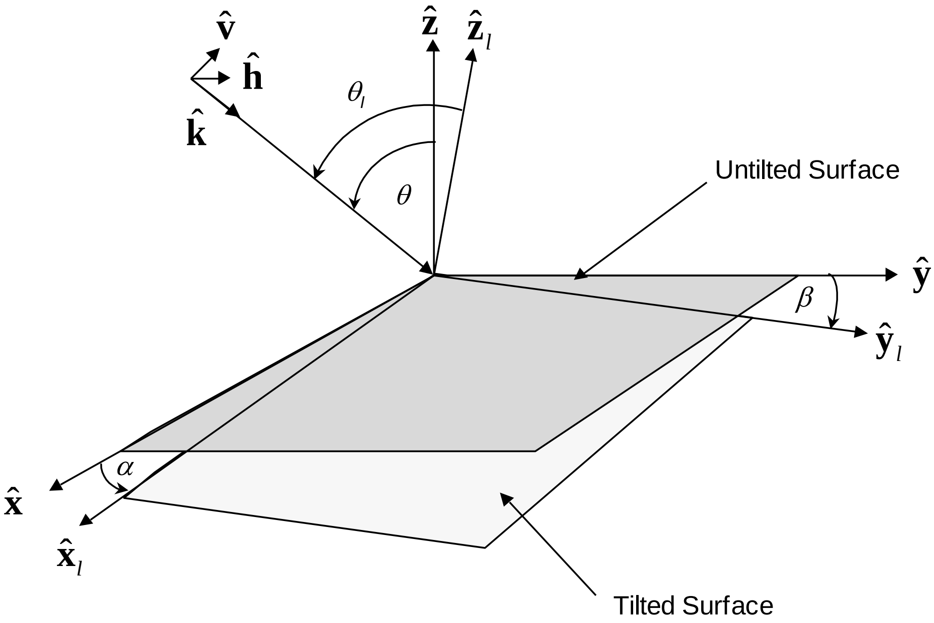

Fig. A-1. Scattering geometry for a tilted surface. The surface is tilted by an angle α in the range (or cross-track) direction and by an angle β in the azimuth(or along-track) direction. The radar measures the scattering coefficients in the global coordinate system, whereas scattering models predict the scattering coefficients in the local coordinate system.

the azimuth direction; that is, in the $y-z$ plane. We define positive values of β as those that tilt the surface in a clockwise direction. The local normal to the surface can then be written as

$$\hat{\mathbf{n}}_l = \frac{\tan\alpha\hat{\mathbf{x}} + \tan\beta\hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{\tan^2\alpha + \tan^2\beta + 1}} = \frac{h_x\hat{\mathbf{x}} + h_y\hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{1 + h_x^2 + h_y^2}}. \tag{A.4}$$

with h_x and h_y the slopes in the x and y directions, respectively. Therefore, the z-axis unit vector in the local coordinate system is

$$\hat{\mathbf{z}}_l = \frac{h_x\hat{\mathbf{x}} + h_y\hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{1 + h_x^2 + h_y^2}}. \tag{A.5}$$

Using Eq. (A.5) in Eq. (A.2), we find the local incidence angle as

$$\cos\theta_l = \frac{h_x\sin\theta + \cos\theta}{\sqrt{1 + h_x^2 + h_y^2}}. \tag{A.6}$$

The radar coordinates expressed in the local coordinate system are needed to calculate the scattering cross-section in the global coordinate system. The scattering matrix in the local coordinate system is calculated assuming

$$\begin{aligned} \hat{\mathbf{h}}_l &= -\hat{\mathbf{k}} \times \hat{\mathbf{z}}_l \Big/ \left| \hat{\mathbf{k}} \times \hat{\mathbf{z}}_l \right| \\ \hat{\mathbf{v}}_l &= -\hat{\mathbf{h}}_l \times \hat{\mathbf{k}} \Big/ \left| \hat{\mathbf{h}}_l \times \hat{\mathbf{k}} \right| \end{aligned} . \tag{A.7}$$

From Eq. (A.1) and Eq. (A.5) we find that

$$\hat{\mathbf{h}}_l = \frac{-h_y \cos\theta \hat{\mathbf{x}} - (\sin\theta - h_x \cos\theta)\hat{\mathbf{y}} + h_y \sin\theta \hat{\mathbf{z}}}{\sqrt{h_y^2 + (\sin\theta - h_x \cos\theta)^2}} . \tag{A.8}$$

Using Eq. (A.8) and Eq. (A.1) in Eq. (A.7), we find

$$\hat{\mathbf{v}}_l = \frac{\cos\theta (h_x \cos\theta - \sin\theta)\hat{\mathbf{x}} + h_y \hat{\mathbf{y}} - \sin\theta (h_x \cos\theta - \sin\theta)\hat{\mathbf{z}}}{\sqrt{h_y^2 + (\sin\theta - h_x \cos\theta)^2}} . \tag{A.9}$$

Note that in the absence of tilts, Eq. (A.9) and Eq. (A.8) reduce to Eq. (A.1).

The scattering matrix in the local coordinate system is defined in these coordinates. We can now calculate the relationship between the scattering matrix measured in the local coordinates, which is usually what is predicted by models, and the one measured in the global coordinates as seen by the radar.

Note that for the purpose of calculating the scattering cross-sections in the global coordinate system, we do not need to know the exact expressions for the local (x_l, y_l, z_l) coordinates; they were already taken into account during the modeling process when the scattering model was originally formulated. All we need to do here is to transform from the local (h_l, v_l, k) coordinate system to the global (h, v, k) system. Let $\vec{E}^t$ be the electric field transmitted by the radar. Then

$$\vec{E}^t = E_h^t \hat{\mathbf{h}} + E_v^t \hat{\mathbf{v}} = E_{hl}^t \hat{\mathbf{h}}_l + E_{vl}^t \hat{\mathbf{v}}_l . \tag{A.10}$$

From this, it follows that

$$E_{hl}^t = \vec{E}^t \cdot \hat{\mathbf{h}}_l = E_h^t \left(\hat{\mathbf{h}} \cdot \hat{\mathbf{h}}_l \right) + E_v^t \left(\hat{\mathbf{v}} \cdot \hat{\mathbf{h}}_l \right) \tag{A.11}$$

and

$$E_{vl}^{t} = \vec{E}^{t} \cdot \hat{\mathbf{v}}_l = E_h^t \left(\hat{\mathbf{h}} \cdot \hat{\mathbf{v}}_l \right) + E_v^t \left(\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_l \right). \tag{A.12}$$

Thus, we can write

$$\vec{E}_l^t = [\mathbf{T}] \vec{E}^t, \tag{A.13}$$

where the transformation matrix is given by

$$[\mathbf{T}] = \begin{pmatrix} \hat{\mathbf{h}} \cdot \hat{\mathbf{h}}_l & \hat{\mathbf{v}} \cdot \hat{\mathbf{h}}_l \\ \hat{\mathbf{h}} \cdot \hat{\mathbf{v}}_l & \hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_l \end{pmatrix}. \tag{A.14}$$

Similarly,

$$E_h^t = \vec{E}_l^t \cdot \hat{\mathbf{h}} = E_{hl}^t \left(\hat{\mathbf{h}} \cdot \hat{\mathbf{h}}_l \right) + E_v^t \left(\hat{\mathbf{h}} \cdot \hat{\mathbf{v}}_l \right) \tag{A.15}$$

and

$$E_v^t = \vec{E}_l^t \cdot \hat{\mathbf{v}} = E_{hl}^t \left(\hat{\mathbf{v}} \cdot \hat{\mathbf{h}}_l \right) + E_v^t \left(\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_l \right). \tag{A.16}$$

Thus, we can write the transformation from the local to the global coordinates as

$$\vec{E}^t = \left[\tilde{\mathbf{T}} \right] \vec{E}_l^t, \tag{A.17}$$

where the ~ character means transpose.

Now we are ready to derive an expression for the scattering matrix in the global coordinate system. First, we transform the electric field transmitted by the radar into the local coordinate system. This is defined by Eq. (A.13) and Eq. (A.14) above. Next, the scattered electric field in the local coordinate system is given by

$$\vec{E}_l^{sc} = \left[\mathbf{S}_l \left(\theta_l \right) \right] \vec{E}_l^t = \left[\mathbf{S}_l \left(\theta_l \right) \right] [\mathbf{T}] \vec{E}^t. \tag{A.18}$$

The addition of the local angle of incidence to the local scattering matrix is a reminder that the appropriate angle of incidence should be calculated using Eq. (A.6). After transforming this scattered electric field to the global coordinate system, we find

$$\vec{E}^{sc} = \left[\tilde{\mathbf{T}} \right] \vec{E}_l^{sc} = \left[\tilde{\mathbf{T}} \right] \left[\mathbf{S}_l \left(\theta_l \right) \right] [\mathbf{T}] \vec{E}^t. \tag{A.19}$$

Therefore, the scattering matrices in the global and local coordinate systems are related by

$$\left[\mathbf{S}(\theta)\right]=\left[\tilde{\mathbf{T}}\right]\left[\mathbf{S}_l\left(\theta_l\right)\right]\left[\mathbf{T}\right]. \tag{A.20}$$

To understand the effect of this transformation, we now need explicit expressions for the elements of the transformation matrix. These are

$$\left[\mathbf{T}\right]=\frac{1}{\sqrt{h_y^2+u^2}}\begin{pmatrix} u & h_y \\ -h_y & u \end{pmatrix}; \quad u=\sin\theta-h_x\cos\theta . \tag{A.21}$$

These expressions show that the scattering matrix in the global coordinate system is related to the one in the local coordinate system through a rotation angle that is a function of three parameters: the azimuth slope h_y of the pixel, the range slope h_x of the pixel, and the angle of incidence θ for the pixel. This rotation angle is

$$\tan\varphi=\frac{h_y}{u} . \tag{A.22}$$

We can write Eq. (A.21) as

$$\left[\mathbf{T}\right]=\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{A.23}$$

To be really useful in our studies, we need to know how the covariance matrix in the global coordinate system is related to the one in the local coordinate system. This can be derived as follows. First, we note that we can write

$$\vec{S}=\begin{pmatrix} S_{hh} \\ \sqrt{2}S_{hv} \\ S_{vv} \end{pmatrix}=\left[\mathbf{Q}\right]\vec{S}_l \tag{A.24}$$

with

$$\left[\mathbf{Q}\right]=\frac{1}{2}\begin{pmatrix} 1+\cos 2\varphi & -\sqrt{2}\sin 2\varphi & 1-\cos 2\varphi \\ \sqrt{2}\sin 2\varphi & 2\cos 2\varphi & -\sqrt{2}\sin 2\varphi \\ 1-\cos 2\varphi & \sqrt{2}\sin 2\varphi & 1+\cos 2\varphi \end{pmatrix}. \tag{A.25}$$

The covariance matrix is defined as

$$\left[\Sigma\right]=\vec{S}\tilde{\vec{S}}^*=\left[\mathbf{Q}\right]\vec{S}_l\tilde{\vec{S}}_l^*\left[\tilde{\mathbf{Q}}\right]^*=\left[\mathbf{Q}\right]\left[\Sigma_l\right]\left[\tilde{\mathbf{Q}}\right]. \tag{A.26}$$

This expression allows us to calculate the covariance matrix in the global coordinate system once we know the covariance matrix in the local coordinate system.