SECTION 6

SPACE-FIXED POSITION, VELOCITY, AND ACCELERATION VECTORS OF A LANDED SPACECRAFT RELATIVE TO CENTER OF MASS OF PLANET, PLANETARY SYSTEM, OR THE MOON

Contents

6.1 INTRODUCTION

This section gives the formulation for the space-fixed position, velocity, and acceleration vectors of a landed spacecraft. The landed spacecraft may be on the surface of a planet, an asteroid, a comet, the Moon, or a satellite of an outer planet. If the lander is on the surface of Mercury, Venus, an asteroid, a comet, or the Moon, the space-fixed vectors will be with respect to the center of mass of that body. If the lander is on the planet or planetary satellite of one of the outer planet systems, the space-fixed vectors will be with respect to the center of mass of the planetary system. The space-fixed position, velocity, and acceleration vectors of the lander are referred to the celestial reference frame defined by the planetary ephemeris (the planetary ephemeris frame, PEF) (see Section 3.1.1).

Section 6.2 gives the formulation for the body-fixed position vector $\mathbf{r}_b$ of a landed spacecraft on body B. The rectangular components of this vector are referred to the true pole, prime meridian, and equator of date. Section 6.3 gives the formulation for the body-fixed to space-fixed transformation matrix T_B (for body B) and its first and second time derivatives with respect to coordinate time ET.

Section 6.4.1 uses $\mathbf{r}_b$ and T_B and its time derivatives to calculate the space-fixed position, velocity, and acceleration vectors of the landed spacecraft relative to the center of mass of body B. If body B is the planet or a planetary satellite of one of the outer planet systems, the satellite ephemeris is interpolated for the position, velocity, and acceleration vectors of body B relative to the center of mass of the planetary system. Adding these two sets of vectors (Section 6.4.2) gives the position, velocity, and acceleration vectors of the landed spacecraft relative to the center of mass of the planetary system.

Section 6.5 gives the formulation for calculating the partial derivatives of the space-fixed position vector of the landed spacecraft with respect to solve-for parameters. There are three groups of these parameters. The first group consists of the three body-fixed spherical or cylindrical coordinates of the landed spacecraft. The second group consists of the six solve-for parameters of the body-fixed to space-fixed transformation matrix T_B. If the lander is resting on the planet or a planetary satellite of a planetary system, the third group consists of the solve-for parameters of the satellite ephemeris for this planetary system.

The time argument for calculating the space-fixed position, velocity, and acceleration vectors of the landed spacecraft is coordinate time ET of the Solar-System barycentric space-time frame of reference. In the spacecraft light-time solution, the time argument will be the reflection time or transmission time t_2(ET) in coordinate time ET at the landed spacecraft.

6.2 BODY-FIXED POSITION VECTOR OF LANDED SPACECRAFT

The body-fixed position vector $\mathbf{r}_b$ of the landed spacecraft with rectangular components referred to the true pole, prime meridian, and equator of date is given by the first term of Eq. (5–1) without the scale factor α. For cylindrical body-fixed coordinates u, v, and λ, $\mathbf{r}_b$ is given by Eq. (5–2). For spherical body-fixed coordinates r, ϕ, and λ, $\mathbf{r}_b$ is given by Eq. (5–3).

6.3 BODY-FIXED TO SPACE-FIXED TRANSFORMATION MATRIX T_B AND ITS TIME DERIVATIVES

This section gives the formulation for the body-fixed to space-fixed transformation matrix T_B and its first and second time derivatives with respect to coordinate time ET. This rotation matrix is used for all bodies of the Solar System except the Earth. Subsection 6.3.1 gives the high-level equations for calculating T_B and its time derivatives. These matrices are a function of three angles and their time derivatives. The angles $\alpha + \Delta\alpha$ and $\delta + \Delta\delta$ are the right ascension and declination of the body's true north pole of date relative to the mean Earth equator and equinox of J2000. The angle $W + \Delta W$ is measured along the body's true equator in the positive sense with respect to the body's true north pole (*i.e.*, in an easterly direction on the body's surface) from the ascending node of the body's true equator on the mean Earth equator of J2000 to the body's prime (*i.e.*, 0°) meridian. This geometry is shown in Fig. 1 of Davies *et al.* (1996). Subsection 6.3.2 gives the formulation for calculating the angles α, δ, and W. The linear terms in α and δ represent precession. The linear term in W is the body's rotation rate. Expressions are also given for the time derivatives of these three angles. The effects of nutation on the angles α, δ, and W are contained in the separate terms $\Delta\alpha$, $\Delta\delta$, and ΔW. The formulation for calculating these angles and their time derivatives is given in Subsection 6.3.3.

6.3.1 HIGH-LEVEL EQUATIONS FOR T_B AND ITS TIME DERIVATIVES

The body-fixed to space-fixed transformation matrix T_B is used to transform the body-fixed position vector $\mathbf{r}_b$ of a landed spacecraft to the corresponding space-fixed position vector $\mathbf{r}_L^B$ of the landed spacecraft (L) relative to the center of mass of body B:

$$\mathbf{r}_L^B = T_B\, \mathbf{r}_b \qquad \text{km} \qquad (6\text{–}1)$$

where

$$T_B = A^T \qquad (6\text{–}2)$$

The matrix A is computed as the product of three coordinate system rotations:

$$A = R_z\left(W + \Delta W\right) R_x\left(\tfrac{\pi}{2} - \delta - \Delta\delta\right) R_z\left(\alpha + \Delta\alpha + \tfrac{\pi}{2}\right) \qquad (6\text{–}3)$$

where the coordinate system rotation matrices are given by Eqs. (5–16) and (5–18). The angles in Eq. (6–3) were defined in Section 6.3. The formulations for computing them are given in Subsections 6.3.2 and 6.3.3. From the transpose of Eq. (6–1), the transformation from space-fixed to body-fixed coordinates of a landed spacecraft is given by:

$$\mathbf{r}_b = T_B{}^T\, \mathbf{r}_L^B = A\, \mathbf{r}_L^B \qquad \text{km} \qquad (6\text{–}4)$$

The space-fixed position, velocity, and acceleration vectors of the landed spacecraft are referred to the celestial reference frame defined by the planetary ephemeris (the planetary ephemeris frame). Since the planetary ephemeris frame can have a slightly different orientation for each planetary ephemeris, the matrix A given by Eq. (6–3) should be post-multiplied by the product $R_x R_y R_z$ of the three frame-tie rotation matrices as was done in Eq. (5–116) for the transpose of the Earth-fixed to space-fixed transformation matrix. The frame-tie rotation matrices have not been added to the transformation matrix T_B used for all bodies other than the Earth because these matrices are considerably less accurate than the matrix T_E used for the Earth. Furthermore, if the user desires to obtain accurate fits to tracking data obtained from a landed spacecraft, he can use one of the later DE400 series planetary ephemerides which are on the radio frame to high accuracy. For these ephemerides, the frame-tie rotation angles are zero.

From Eqs. (6–2) and (6–3), the derivative of T_B with respect to coordinate time ET is given by:

$$\dot{T}_B = \dot{A}^T \tag{6–5}$$

where

$$\begin{aligned}\dot{A} = &\frac{dR_z(W+\Delta W)}{d(W+\Delta W)} R_x\left(\tfrac{\pi}{2}-\delta-\Delta\delta\right) R_z\left(\alpha+\Delta\alpha+\tfrac{\pi}{2}\right)\left[\dot{W}+(\Delta W)^{\cdot}\right] \\ &- R_z(W+\Delta W)\frac{dR_x\left(\tfrac{\pi}{2}-\delta-\Delta\delta\right)}{d\left(\tfrac{\pi}{2}-\delta-\Delta\delta\right)} R_z\left(\alpha+\Delta\alpha+\tfrac{\pi}{2}\right)\left[\dot{\delta}+(\Delta\delta)^{\cdot}\right] \qquad \text{rad/s} \\ &+ R_z(W+\Delta W) R_x\left(\tfrac{\pi}{2}-\delta-\Delta\delta\right)\frac{dR_z\left(\alpha+\Delta\alpha+\tfrac{\pi}{2}\right)}{d\left(\alpha+\Delta\alpha+\tfrac{\pi}{2}\right)}\left[\dot{\alpha}+(\Delta\alpha)^{\cdot}\right]\end{aligned} \tag{6–6}$$

where the coordinate system rotation matrices and their derivatives with respect to the rotation angles are given by Eqs. (5–16) and (5–18). The time derivatives $\dot{\alpha}$, $\dot{\delta}$, and $\dot{W}$ of the angles α, δ, and W are computed from the formulation given in Subsection 6.3.2. The time derivatives $(\Delta\alpha)^{\cdot}$, $(\Delta\delta)^{\cdot}$, and $(\Delta W)^{\cdot}$ of the angles $\Delta\alpha$, $\Delta\delta$, and ΔW are computed from the formulation given in Subsection 6.3.3.

From Eqs. (6–2), (6–3), and (5–18), the second time derivative of T_B can be calculated to sufficient accurcy by calculating:

$$\ddot{T}_B = -T_B \left[\dot{W} + (\Delta W)^{\cdot} \right]^2 \qquad \text{rad/s}^2 \qquad (6\text{–}7)$$

and then setting column three of this 3 x 3 matrix to zero.

6.3.2 EXPRESSIONS FOR α, δ, AND W AND THEIR TIME DERIVATIVES

The expressions for $\alpha + \Delta\alpha$, $\delta + \Delta\delta$, and $W + \Delta W$ for the Sun and the planets are given in Table I of Davies *et al*. (1996). The corresponding expressions for the planetary satellites are given in Table II of this reference. The angles α, δ, and W are polynomials in time. The angles $\Delta\alpha$, $\Delta\delta$, and ΔW contain periodic terms only. The angles α, δ, and W are represented by the following linear or quadratic functions of time in the ODP:

$$\alpha = \left[\alpha_o + \dot{\alpha}_o (T - T_o) \right] / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}8)$$

$$\delta = \left[\delta_o + \dot{\delta}_o (T - T_o) \right] / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}9)$$

$$W = \left[W_o + \dot{W}_o (d - d_o) + Q (T - T_o)^2 \right] / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}10)$$

where T is Julian centuries of coordinate time ET past J2000, calculated from Eq. (5–65). The variable d is days of coordinate time ET past J2000, which is calculated from:

$$d = \frac{\text{ET}}{86400} \qquad (6\text{–}11)$$

where ET is seconds of coordinate time past J2000. The terms $\dot{\alpha}_o$ and $\dot{\delta}_o$ represent precession of the body's true north pole, and $\dot{W}_o$ is the nominal rotation rate of the body. The constant and linear coefficients in Eqs. (6–8) to

(6–10) can be estimated at a user-input epoch, which is converted to T_o Julian centuries past J2000 and d_o days past J2000. Numerical values of the coefficients in Eqs. (6–8) to (6–10) at the epoch J2000 (*i.e.*, $T_o = d_o = 0$) can be obtained from Tables I and II of Davies *et al*. (1996). These coefficients are in the units of degrees, degrees per Julian century or day, and degrees per Julian century squared. The constant DEGR = 57.295,779,513,082,3209 degrees per radian.

From Tables I and II of Davies *et al*. (1996), the only bodies that have a non-zero quadratic coefficient Q in Eq. (6–10) for W are the Moon and the satellites of Mars. For Phobos and Deimos, Q is given in degrees per Julian century squared as shown in Eq. (6–10). However, for the Moon, Q is given as -1.4×10^{-12} degrees per day squared. It can be converted to degrees per Julian century squared for use in Eq. (6–10) by multiplying by the square of 36525, which gives -1.8677×10^{-3} degrees per Julian century squared.

If the user desires to estimate the constant and linear coefficients of Eqs. (6–8) to (6–10) at a user-input epoch, the coefficients obtained from Tables I and II of Davies *et al*. (1996), which apply at the epoch J2000, must be converted to values at the user-input epoch. The constant coefficients in these equations must be replaced with:

$$\alpha_o + \dot{\alpha}_o T_o$$
$$\delta_o + \dot{\delta}_o T_o$$
$$W_o + \dot{W}_o d_o + Q T_o^2$$

and $\dot{W}_o$ must be replaced with:

$$\dot{W}_o + \frac{2 Q T_o}{36525}$$

The coefficients $\dot{\alpha}_o$, $\dot{\delta}_o$, and Q and are not changed because they are constant.

From Eqs. (6–8) to (6–10), the time derivatives of α, δ, and W in radians per second of coordinate time ET are given by:

$$\dot{\alpha} = \frac{\dot{\alpha}_{\mathrm{o}}}{86400 \times 36525 \times \mathrm{DEGR}} \qquad \mathrm{rad/s} \qquad (6\text{–}12)$$

$$\dot{\delta} = \frac{\dot{\delta}_{\mathrm{o}}}{86400 \times 36525 \times \mathrm{DEGR}} \qquad \mathrm{rad/s} \qquad (6\text{–}13)$$

$$\dot{W} = \frac{1}{86400 \times \mathrm{DEGR}} \left[\dot{W}_{\mathrm{o}} + \frac{2Q\left(T - T_{\mathrm{o}}\right)}{36525} \right] \qquad \mathrm{rad/s} \qquad (6\text{–}14)$$

6.3.3 EXPRESSIONS FOR $\Delta\alpha$, $\Delta\delta$, AND ΔW AND THEIR TIME DERIVATIVES

The angles $\Delta\alpha$, $\Delta\delta$, and ΔW are represented by the following periodic functions of time in the ODP:

$$\Delta\alpha,\ \Delta\delta,\ \Delta W = \sum_{i=1}^{n} \frac{C_i}{\mathrm{DEGR}} \begin{pmatrix} \sin \\ \cos \end{pmatrix} A_i \qquad \mathrm{rad} \qquad (6\text{–}15)$$

The expressions for $\Delta\alpha$, $\Delta\delta$, and ΔW for the satellites of the Earth, Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto may be obtained from Table II of Davies *et al*. (1996). The expressions used for the planet Neptune may be obtained from Table I of this reference. Each satellite (or planet) has separate coefficients C_i (in degrees) for each of the angles $\Delta\alpha$, $\Delta\delta$, and ΔW. Each planetary system has one set of polynomials for calculating the arguments A_1 to A_n. However, each satellite (or the planet) of a planetary system can use some or all of the arguments A_1 to A_n for that system plus integer multiples of these arguments. In the input program GIN of the ODP, the user must input the coefficients (specified below) of each of the polynomials A_1 to A_n used for each satellite (or planet) and the corresponding coefficients C_1 to C_n used for each of the three angles $\Delta\alpha$, $\Delta\delta$, and ΔW. The angles $\Delta\alpha$ and ΔW are computed from sines of A_i while $\Delta\delta$ is computed from cosines of A_i.

For the Moon and satellites of Mars, the arguments A_i in radians are computed from:

$$A_i = \left(A_{i_0} + A_{i_1} d + A_{i_2} T^2 \right) / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}16)$$

where the coefficients on the right-hand side are in units of degrees. For the satellites of Jupiter, Saturn, Uranus, Neptune, and Pluto,

$$A_i = \left(A_{i_0} + A_{i_1} T \right) / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}17)$$

The coefficients for A_1 to A_n for the planetary systems Earth through Pluto are given in Table II of Davies *et al*. (1996).

The expression for ΔW for Deimos in Table II of Davies *et al*. (1996) contains the term:

$$0^\circ.19 \cos M3 \qquad (6\text{–}18)$$

where $M3$ is A_3 for Mars which is given by:

$$M3 = \left(53^\circ.47 - 0^\circ.0181510\, d \right) / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}19)$$

Also, note that Mars uses the arguments A_i equal to $M1$, $M2$, and $M3$. In order to make the term (6–18) consistent with Eq. (6–15), we must change the cosine in this term to a sine. This can be accomplished by defining $M4$ to be equal to $M3$ plus $\pi/2$ radians:

$$M4 = \left(143^\circ.47 - 0^\circ.0181510\, d \right) / \text{DEGR} \qquad \text{rad} \qquad (6\text{–}20)$$

Then the term (6–18) can be replaced with the term:

$$0^\circ.19 \sin M4 \qquad (6\text{–}21)$$

which is consistent with Eq. (6–15).

From Eq. (6–15), the time derivatives of $\Delta\alpha$, $\Delta\delta$, and ΔW in radians per second of coordinate time ET are given by:

$$(\Delta\alpha)^{\cdot},\ (\Delta\delta)^{\cdot},\ (\Delta W)^{\cdot} = \sum_{i=1}^{n} \frac{C_i\,\dot{A}_i}{\text{DEGR}} \begin{pmatrix} \cos \\ -\sin \end{pmatrix} A_i \qquad \text{rad/s} \qquad (6\text{–}22)$$

where $(\Delta\alpha)^{\cdot}$ and $(\Delta W)^{\cdot}$ are computed from cosines of A_i and $(\Delta\delta)^{\cdot}$ is computed from the negative of sines of A_i. From Eq. (6–16), the time derivatives of the arguments A_i for the Moon and satellites of Mars in radians per second are given by:

$$\dot{A}_i = \frac{1}{86400 \times \text{DEGR}} \left(A_{i_1} + \frac{2\,A_{i_2}\,T}{36525} \right) \qquad \text{rad/s} \qquad (6\text{–}23)$$

From Eq. (6–17), the time derivatives of the arguments A_i for the satellites of Jupiter, Saturn, Uranus, Neptune, and Pluto in radians per second are given by:

$$\dot{A}_i = \frac{A_{i_1}}{86400 \times 36525 \times \text{DEGR}} \qquad \text{rad/s} \qquad (6\text{–}24)$$

6.4 SPACE-FIXED POSITION, VELOCITY, AND ACCELERATION VECTORS OF LANDED SPACECRAFT

6.4.1 SPACE-FIXED VECTORS RELATIVE TO LANDER BODY B

6.4.1.1 Rotation From Body-Fixed to Space-Fixed Coordinates

The transformation of the body-fixed position vector $\mathbf{r}_{\text{b}}$ of a landed spacecraft on body B to the corresponding space-fixed position vector $\mathbf{r}_{\text{L}}^{\text{B}}$ of the landed spacecraft relative to the center of mass of body B is given by Eqs. (6–1) through (6–3). Since $\mathbf{r}_{\text{b}}$ is fixed, the space-fixed velocity and acceleration vectors of the landed spacecraft relative to body B can be computed from the following derivatives of Eq. (6–1) with respect to coordinate time ET:

$$\dot{\mathbf{r}}_{\text{L}}^{\text{B}} = \dot{T}_{\text{B}}\ \mathbf{r}_{\text{b}} \qquad \text{km/s} \qquad (6\text{–}25)$$

$$\ddot{\mathbf{r}}_{\text{L}}^{\text{B}} = \ddot{T}_{\text{B}}\ \mathbf{r}_{\text{b}} \qquad \text{km/s}^2 \qquad (6\text{–}26)$$

where $\dot{T}_\mathrm{B}$ is given by Eqs. (6–5) and (6–6) and $\ddot{T}_\mathrm{B}$ is obtained by evaluating Eq. (6–7) and then setting column three of this 3 x 3 matrix to zero. In these equations, the angles α, δ, and W and $\Delta\alpha$, $\Delta\delta$, and ΔW and their time derivatives are calculated from the formulations given in Sections 6.3.2 and 6.3.3 using coefficients obtained from Davies *et al*. (1996).

6.4.1.2 Transformation of Space-Fixed Position Vector of Lander Relative to Body B From Local Space-Time Frame of Reference of Body B to Solar-System Barycentric Space-Time Frame of Reference

The space-fixed position vector $\mathbf{r}_\mathrm{L}^\mathrm{B}$ of the landed spacecraft L relative to body B calculated from Eq. (6–1) is in the local space-time frame of reference of body B. This vector must be transformed from the local space-time frame of reference of body B to the Solar-System barycentric space-time frame of reference. The equation used when body B is the Earth is Eq. (4–10). Applying this equation to body B gives:

$$\left(\mathbf{r}_\mathrm{L}^\mathrm{B}\right)_\mathrm{BC} = \left(1 - \tilde{L}_\mathrm{B} - \frac{\gamma U_\mathrm{B}}{c^2}\right)\mathbf{r}_\mathrm{L}^\mathrm{B} - \frac{1}{2c^2}\left(\mathbf{V}_\mathrm{B} \cdot \mathbf{r}_\mathrm{L}^\mathrm{B}\right)\mathbf{V}_\mathrm{B} \qquad \mathrm{km} \qquad (6\text{–}27)$$

where $\left(\mathbf{r}_\mathrm{L}^\mathrm{B}\right)_\mathrm{BC}$ is the space-fixed position vector of the landed spacecraft L relative to body B in the Solar-System barycentric space-time frame of reference. The gravitational potential U_B at body B is calculated from Eq. (2–17) where i = B (body B). The quantity $\mathbf{V}_\mathrm{B}$ is the velocity vector of body B relative to the Solar-System barycenter. The quantity $\tilde{L}_\mathrm{B}$ is analogous to $\tilde{L}$, which applies at the Earth. From Eq. (4–7), the value of $\tilde{L}_\mathrm{B}$ at body B is the value of the constant L defined by Eq. (2–22) at the landed spacecraft on body B in the Solar-System barycentric space-time frame of reference minus the corresponding value L_B defined by Eq. (2–22) at the landed spacecraft in the local space-time frame of reference of body B. The analytical expression and numerical value of $\tilde{L}$ for the Earth are given by Eqs. (4–16) and (4–17). Eq. (4–16) is L given by Eq. (4–12) minus L_GC given by Eq. (4–14). We need expressions for $\tilde{L}_\mathrm{B}$ for each body B where we expect to have a landed spacecraft. The obvious first candidates are Mars and the Moon.

Eq. (4–16) for $\tilde{L}$ at the Earth changes to the following expression for $\tilde{L}_{\mathrm{Ma}}$ at Mars:

$$\tilde{L}_{\mathrm{Ma}} = \frac{1}{c^2\,AU}\left[\frac{\mu_{\mathrm{S}} + \mu_{\mathrm{Me}} + \mu_{\mathrm{V}} + \mu_{\mathrm{E}} + \mu_{\mathrm{M}}}{a_{\mathrm{Ma}}} + \frac{\mu_{\mathrm{J}}}{a_{\mathrm{J}}} + \frac{\mu_{\mathrm{Sa}}}{a_{\mathrm{Sa}}} + \frac{\mu_{\mathrm{U}}}{a_{\mathrm{U}}} + \frac{\mu_{\mathrm{N}}}{a_{\mathrm{N}}} + \frac{\mu_{\mathrm{Pl}}}{a_{\mathrm{Pl}}} + \frac{\mu_{\mathrm{S}} + \mu_{\mathrm{Ma}}}{2\,a_{\mathrm{Ma}}}\right] \tag{6–28}$$

For an accuracy of 0.01 mm in $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{Ma}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27), all of the "small body" terms in Eq. (6–28) can be deleted, which gives:

$$\tilde{L}_{\mathrm{Ma}} = \frac{3\,\mu_{\mathrm{S}}}{2\,c^2\,AU\,a_{\mathrm{Ma}}} \tag{6–29}$$

Inserting numerical values from Section 4.3.1.2 gives:

$$\tilde{L}_{\mathrm{Ma}} = 0.9717 \times 10^{-8} \tag{6–30}$$

From Table 15.8 on p. 706 of the *Explanatory Supplement* (1992), the equatorial radius of Mars is 3397 km. The effect of a change of 1 in the last digit of $\tilde{L}_{\mathrm{Ma}}$ given by Eq. (6–30) on $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{Ma}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27) is 0.003 mm. The effect of $\tilde{L}_{\mathrm{Ma}}$ on $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{Ma}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27) is about 3.3 cm. The first term of Eq. (6–27) reduces the radius of Mars at the lander by about 5.5 cm in the Solar-System barycentric space-time frame of reference.

Eq. (4–16) for $\tilde{L}$ at the Earth changes to the following approximate expression for $\tilde{L}_{\mathrm{M}}$ at the Moon:

$$\tilde{L}_{\mathrm{M}} = \frac{1}{c^2}\left[\frac{1}{AU}\left(\frac{\mu_{\mathrm{S}} + \mu_{\mathrm{Me}} + \mu_{\mathrm{V}}}{a_{\mathrm{B}}} + \frac{\mu_{\mathrm{Ma}}}{a_{\mathrm{Ma}}} + \frac{\mu_{\mathrm{J}}}{a_{\mathrm{J}}} + \frac{\mu_{\mathrm{Sa}}}{a_{\mathrm{Sa}}} + \frac{\mu_{\mathrm{U}}}{a_{\mathrm{U}}} + \frac{\mu_{\mathrm{N}}}{a_{\mathrm{N}}} + \frac{\mu_{\mathrm{Pl}}}{a_{\mathrm{Pl}}}\right) + \frac{\mu_{\mathrm{E}}}{a_{\mathrm{M}}} + \frac{\mu_{\mathrm{S}} + \mu_{\mathrm{E}} + \mu_{\mathrm{M}}}{2\,AU\,a_{\mathrm{B}}} + \frac{\mu_{\mathrm{E}}}{2\,a_{\mathrm{M}}}\right] \qquad (6\text{–}31)$$

where a_{B} is the semi-major axis of the heliocentric orbit of the Earth-Moon barycenter B in astronomical units. The largest of the "small body" terms in this equation is the gravitational potential at the Moon due to the Earth multiplied by 3/2. It changes $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{M}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27) by about 0.03 mm, which can be ignored. Hence, all of the "small body" terms in Eq. (6–31) can be ignored which gives:

$$\tilde{L}_{\mathrm{M}} = \frac{3\,\mu_{\mathrm{S}}}{2\,c^2\,AU\,a_{\mathrm{B}}} \qquad (6\text{–}32)$$

Inserting numerical values from Section 4.3.1.2 gives:

$$\tilde{L}_{\mathrm{M}} = 1.4806 \times 10^{-8} \qquad (6\text{–}33)$$

From Table 15.8 on p. 706 of the *Explanatory Supplement* (1992), the equatorial radius of the Moon is 1738 km. The effect of a change of 1 in the last digit of $\tilde{L}_{\mathrm{M}}$ given by Eq. (6–33) on $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{M}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27) is 0.002 mm. The effect of $\tilde{L}_{\mathrm{M}}$ on $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{M}}\right)_{\mathrm{BC}}$ computed from Eq. (6–27) is about 2.6 cm. The first term of Eq. (6–27) reduces the radius of the Moon at the lander by about 4.3 cm in the Solar-System barycentric space-time frame of reference.

The general expression for $\tilde{L}_{\mathrm{B}}$ for a lander on any planet, asteroid, or comet is the generalization of Eq. (6–29):

$$\tilde{L}_{\mathrm{planet}} = \frac{3\,\mu_{\mathrm{S}}}{2\,c^2\,AU\,a_{\mathrm{planet}}} \qquad (6\text{–}34)$$

where a_{planet} is the semi-major axis of the heliocentric orbit of the planet, asteroid, or comet in astronomical units. For a lander on a planetary satellite,

$$\tilde{L}_{\text{satellite}} = \frac{3\mu_{\text{S}}}{2c^2 AU\, a_{\text{planet}}} + \frac{3\mu_{\text{planet}}}{2c^2 a_{\text{satellite}}} \tag{6–35}$$

where a_{planet} is defined above, μ_{planet} is the gravitational constant of the planet, and $a_{\text{satellite}}$ is the semi-major axis of the orbit of the planetary satellite in kilometers. For a lander on the Moon, the second term of this expression is included in Eq. (6–31) but is ignored in Eq. (6–32).

In Eq. (6–27), the gravitational potential U_{B} at the lander body B should include the term due to the Sun plus the term due to a planet if the lander is resting on a satellite of the planet. Note that the latter term is ignored for a lunar lander. If the lander body B is Mercury, Venus, the Moon, an asteroid, or a comet, interpolate the planetary ephemeris (plus the small-body ephemeris of the asteroid or comet) for the position vector $\mathbf{r}_{\text{B}}^{\text{S}}$ from the Sun to body B as described in Section 3.1.2.1. If the lander body B is the planet or a satellite of one of the outer planet systems, interpolate the planetary ephemeris for the position vector $\mathbf{r}_{\text{P}}^{\text{S}}$ from the Sun to the center of mass P of the planetary system and interpolate the satellite ephemeris for the position vector $\mathbf{r}_{\text{B}}^{\text{P}}$ of the lander body B relative to the center of mass P of the planetary system as described in Section 3.2.2.1. The position vector from the Sun to the lander body B is given by:

$$\mathbf{r}_{\text{B}}^{\text{S}} = \mathbf{r}_{\text{P}}^{\text{S}} + \mathbf{r}_{\text{B}}^{\text{P}} \tag{6–36}$$

For a lander on any body B, the distance from body B to the Sun is given by the magnitude of the position vector $\mathbf{r}_{\text{B}}^{\text{S}}$:

$$r_{\text{BS}} = \left|\mathbf{r}_{\text{B}}^{\text{S}}\right| \tag{6–37}$$

If the lander body B is a satellite of one of the outer planet systems, interpolate the satellite ephemeris as described in Section 3.2.2.1 for the position vectors of the lander body B and the planet 0 relative to the center of mass P of the planetary system and calculate the position vector from the lander body B to the planet 0:

$$\mathbf{r}_0^{\mathrm{B}} = \mathbf{r}_0^{\mathrm{P}} - \mathbf{r}_{\mathrm{B}}^{\mathrm{P}} \tag{6–38}$$

The distance from the satellite B that the lander is resting upon to the planet 0 is the magnitude of the position vector $\mathbf{r}_0^{\mathrm{B}}$:

$$r_{\mathrm{B0}} = \left|\mathbf{r}_0^{\mathrm{B}}\right| \tag{6–39}$$

If the landed spacecraft is on the Moon, any planet, an asteroid, or a comet, the gravitational potential U_{B} at the lander body B is given to sufficient accuracy by:

$$U_{\mathrm{B}} = \frac{\mu_{\mathrm{S}}}{r_{\mathrm{BS}}} \tag{6–40}$$

where μ_{S} is the gravitational constant of the Sun obtained from the planetary ephemeris and r_{BS} is given by Eq. (6–37). If the landed spacecraft is on a satellite of one of the outer planet systems, the gravitational potential U_{B} at the lander body B is given to sufficient accuracy by:

$$U_{\mathrm{B}} = \frac{\mu_{\mathrm{S}}}{r_{\mathrm{BS}}} + \frac{\mu_0}{r_{\mathrm{B0}}} \tag{6–41}$$

where μ_0 is the gravitational constant of the planet obtained from the satellite ephemeris as described in Section 3.2.2.1 and r_{B0} is given by Eq. (6–39).

In Eq. (6–27), $\mathbf{V}_{\mathrm{B}}$ is the velocity vector of the lander body B relative to the Solar-System barycenter. For a landed spacecraft on Mercury, Venus, the Moon, an asteroid, or a comet, interpolate the planetary ephemeris (plus the small-body ephemeris of the asteroid or comet) for the velocity vector $\dot{\mathbf{r}}_{\mathrm{B}}^{\mathrm{C}}$ of the lander body B relative to the Solar-System barycenter C. The velocity vector $\mathbf{V}_{\mathrm{B}}$ is given by:

$$\mathbf{V}_B = \dot{\mathbf{r}}_B^C \tag{6–42}$$

For a lander on the planet or planetary satellite of a planetary system, interpolate the satellite ephemeris for that system for the velocity vector $\dot{\mathbf{r}}_B^P$ of the lander body B relative to the center of mass P of the planetary system. Also, interpolate the planetary ephemeris for the velocity vector $\dot{\mathbf{r}}_P^C$ of the center of mass P of the planetary system relative to the Solar-System barycenter C. For this case, the velocity vector $\mathbf{V}_B$ is given by:

$$\mathbf{V}_B = \dot{\mathbf{r}}_P^C + \dot{\mathbf{r}}_B^P \tag{6–43}$$

It is not necessary to transform $\dot{\mathbf{r}}_L^B$ and $\ddot{\mathbf{r}}_L^B$ calculated from Eqs. (6–25) and (6–26) from the local space-time frame of reference of body B to the Solar-System barycentric space-time frame of reference using the first and second time derivatives of Eq. (6–27) with respect to coordinate time in the barycentric frame because the computed values of observed quantities require accurate values of the position vectors of the participants, not accurate values of the velocity and acceleration vectors.

6.4.2 OFFSET FROM CENTER OF MASS OF PLANETARY SYSTEM TO CENTER OF THE LANDER PLANET OR PLANETARY SATELLITE

If the lander body B that the landed spacecraft is resting upon is the planet or a planetary satellite of one of the outer planet systems, then the position, velocity, and acceleration vectors of the lander body B relative to the center of mass P of the planetary system must be interpolated from the satellite ephemeris for that planetary system as described in Section 3.2.2.

If the landed spacecraft is resting upon a satellite or the planet of one of the outer planet systems, the space-fixed position, velocity, and acceleration vectors of the landed spacecraft relative to the center of mass P of the planetary system are computed from the following equations:

$$\mathbf{r}_L^P = \left(\mathbf{r}_L^B\right)_{BC} + \mathbf{r}_B^P \tag{6–44}$$

where $\left(\mathbf{r}_{\mathrm{L}}^{\mathrm{B}}\right)_{\mathrm{BC}}$ is calculated from Eq. (6–27) using $\mathbf{r}_{\mathrm{L}}^{\mathrm{B}}$ calculated from Eq. (6–1). The position vector $\mathbf{r}_{\mathrm{B}}^{\mathrm{P}}$ of the lander body B relative to the center of mass P of the planetary system is obtained from the satellite ephemeris.

$$\dot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{P}} = \dot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{B}} + \dot{\mathbf{r}}_{\mathrm{B}}^{\mathrm{P}} \tag{6–45}$$

where $\dot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{B}}$ is calculated from Eq. (6–25) and $\dot{\mathbf{r}}_{\mathrm{B}}^{\mathrm{P}}$ is obtained from the satellite ephemeris.

$$\ddot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{P}} = \ddot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{B}} + \ddot{\mathbf{r}}_{\mathrm{B}}^{\mathrm{P}} \tag{6–46}$$

where $\ddot{\mathbf{r}}_{\mathrm{L}}^{\mathrm{B}}$ is calculated from Eq. (6–26). In this equation, $\ddot{T}_{\mathrm{B}}$ is obtained by evaluating Eq. (6–7) and then setting column three of this 3 x 3 matrix to zero. The acceleration vector of the lander body B relative to the center of mass P of the planetary system is obtained from the satellite ephemeris.

6.5 PARTIAL DERIVATIVES OF SPACE-FIXED POSITION VECTOR OF LANDED SPACECRAFT

This section gives the formulation for calculating the partial derivatives of the space-fixed position vector of the landed spacecraft with respect to solve-for or consider parameters **q**. Subsection 6.5.1 gives the partial derivatives of the space-fixed position vector $\mathbf{r}_{\mathrm{L}}^{\mathrm{B}}$ of the landed spacecraft L relative to the center of mass of the lander body B with respect to the body-fixed cylindrical or spherical coordinates of the lander. Subsection 6.5.2 gives the partial derivatives of $\mathbf{r}_{\mathrm{L}}^{\mathrm{B}}$ with respect to the six solve-for parameters of the body-fixed to space-fixed transformation matrix T_{B} for body B. If the lander is resting upon the planet or a planetary satellite of one of the outer planet systems, the offset vector $\mathbf{r}_{\mathrm{B}}^{\mathrm{P}}$ from the center of mass P of the planetary system to the lander body B is a function of the solve-for parameters of the satellite ephemeris for this planetary system. The partial derivatives of $\mathbf{r}_{\mathrm{B}}^{\mathrm{P}}$ with respect to the satellite ephemeris parameters are given in Subsection 6.5.3.

6.5.1 CYLINDRICAL OR SPHERICAL COORDINATES OF THE LANDER

From Eq. (6–1), the partial derivatives of the space-fixed position vector of the lander L relative to the center of mass of the lander body B with respect to those parameters $\mathbf{q}$ that affect the body-fixed position vector of the lander are given by:

$$\frac{\partial \mathbf{r}_{\mathrm{L}}^{\mathrm{B}}}{\partial \mathbf{q}} = T_{\mathrm{B}} \frac{\partial \mathbf{r}_{\mathrm{b}}}{\partial \mathbf{q}} \tag{6–47}$$

where, from Section 6.2, the partial derivatives of the body-fixed position vector $\mathbf{r}_{\mathrm{b}}$ of the landed spacecraft with respect to the cylindrical coordinates u, v, and λ of the lander are given by Eqs. (5–203) to (5–205) with the parameter α set to unity. The partial derivatives of $\mathbf{r}_{\mathrm{b}}$ with respect to the spherical coordinates r, ϕ, and λ of the lander are given by Eqs. (5–206) to (5–208) with α set to unity.

6.5.2 PARAMETERS OF THE BODY-FIXED TO SPACE-FIXED TRANSFORMATION MATRIX T_{B}

From Eq. (6–1), the partial derivatives of the space-fixed position vector of the lander L relative to the lander body B with respect to the six solve-for parameters $\mathbf{q}$ of the body-fixed to space-fixed transformation matrix T_{B} for body B are given by:

$$\frac{\partial \mathbf{r}_{\mathrm{L}}^{\mathrm{B}}}{\partial \mathbf{q}} = \frac{\partial T_{\mathrm{B}}}{\partial \mathbf{q}} \mathbf{r}_{\mathrm{b}} \tag{6–48}$$

The solve-for parameters are α_{o}, $\dot{\alpha}_{\mathrm{o}}$, δ_{o}, $\dot{\delta}_{\mathrm{o}}$, W_{o}, and $\dot{W}_{\mathrm{o}}$ of Eqs. (6–8) to (6–10). From Eqs. (6–2), (6–3), and (6–8) to (6–10), the partial derivatives of T_{B} with respect to the six parameters are given by:

$$\frac{\partial T_{\mathrm{B}}}{\partial \alpha_{\mathrm{o}}, \dot{\alpha}_0} = \left[R_{\mathrm{z}}\left(W + \Delta W\right) R_{\mathrm{x}}\left(\tfrac{\pi}{2} - \delta - \Delta\delta\right) \frac{dR_{\mathrm{z}}\left(\alpha + \Delta\alpha + \tfrac{\pi}{2}\right)}{d\left(\alpha + \Delta\alpha + \tfrac{\pi}{2}\right)} \right]^{\mathrm{T}} \frac{\partial \alpha}{\partial \alpha_{\mathrm{o}}, \dot{\alpha}_{\mathrm{o}}} \tag{6–49}$$

where

$$\frac{\partial \alpha}{\partial \alpha_o} = \frac{1}{\text{DEGR}} \tag{6–50}$$

$$\frac{\partial \alpha}{\partial \dot{\alpha}_o} = \frac{T - T_o}{\text{DEGR}} \tag{6–51}$$

$$\frac{\partial T_B}{\partial \delta_o, \dot{\delta}_o} = -\left[R_z\left(W + \Delta W\right) \frac{dR_x\left(\frac{\pi}{2} - \delta - \Delta\delta\right)}{d\left(\frac{\pi}{2} - \delta - \Delta\delta\right)} R_z\left(\alpha + \Delta\alpha + \frac{\pi}{2}\right) \right]^T \frac{\partial \delta}{\partial \delta_o, \dot{\delta}_o} \tag{6–52}$$

where

$$\frac{\partial \delta}{\partial \delta_o} = \frac{1}{\text{DEGR}} \tag{6–53}$$

$$\frac{\partial \delta}{\partial \dot{\delta}_o} = \frac{T - T_o}{\text{DEGR}} \tag{6–54}$$

$$\frac{\partial T_B}{\partial W_o, \dot{W}_o} = \left[\frac{dR_z\left(W + \Delta W\right)}{d\left(W + \Delta W\right)} R_x\left(\frac{\pi}{2} - \delta - \Delta\delta\right) R_z\left(\alpha + \Delta\alpha + \frac{\pi}{2}\right) \right]^T \frac{\partial W}{\partial W_o, \dot{W}_o} \tag{6–55}$$

where

$$\frac{\partial W}{\partial W_o} = \frac{1}{\text{DEGR}} \tag{6–56}$$

$$\frac{\partial W}{\partial \dot{W}_o} = \frac{d - d_o}{\text{DEGR}} \tag{6–57}$$

In these equations, the coordinate system rotation matrices and their derivatives with respect to the coordinate system rotation angles are given by Eqs. (5–16) to (5–18). The quantities $T - T_o$ and $d - d_o$ are discussed after Eq. (6–10).

6.5.3 SATELLITE EPHEMERIS PARAMETERS

If the landed spacecraft is resting upon the planet or a planetary satellite of one of the outer planet systems, the offset position vector $\mathbf{r}_{\mathrm{B}}^{\mathrm{P}}$ of the lander body B relative to the center of mass P of the planetary system is a function of the solve-for parameters of the satellite ephemeris for this planetary system. The partial derivatives of $\mathbf{r}_{\mathrm{B}}^{\mathrm{P}}$ with respect to the satellite ephemeris parameters are obtained by interpolating the satellite partials file for this planetary system (as described in Section 3.2.3) with coordinate time ET of the Solar-System barycentric space-time frame of reference as the argument:

$$\frac{\partial \mathbf{r}_{\mathrm{B}}^{\mathrm{P}}}{\partial \mathbf{q}} \tag{6–58}$$

For a lander on the planet Mars, the magnitude of the offset vector $\mathbf{r}_{\mathrm{Ma}}^{\mathrm{P}}$ is less than 25 cm, and these partial derivatives can be ignored.