the periodicity to simplify the resulting equations, we obtain the result

$$\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{im\phi}\Big[imy^{n+1} - ny^{n-1}(ay + by\cos\phi + c\sin\phi) + \frac{n(n-1)D}{2}y^{n-2}\Big]p(\phi, y)\,d\phi\,dy = 0.$$

This can be written in the form

$$imE(y^{n+1}e^{im\phi}) = nE[(ay^n + by^n\cos\phi + cy^{n-1}\sin\phi)e^{im\phi}] - \frac{n(n-1)}{2}DE(y^{n-2}e^{im\phi}),$$

$$n = 0, 1, \cdots; m = 0, \pm 1, \cdots. \quad (4)$$

These equations in themselves are not sufficient to determine the moments $E(y^n e^{im\phi})$, but they do determine these moments for $n > 0$ if those for $n = 0$ are known. There is apparently no simple way of finding the moments $E(e^{im\phi})$. But, Eqs. (4) are interesting in themselves, and they can be used to derive the useful Eq. (2). This derivation follows.

2. The Main Result

If we let $n = 0$ and $m \neq 0$ in Eq. (4), we obtain

$$E(ye^{im\phi}) = 0. \quad (5)$$

It is evident from Eq. (1) that the behavior of the phase-locked loop is unaffected by the transformation

$$(\phi, y) \rightarrow (-\phi, -y).$$

Thus, for all $m = 0, \pm 1, \cdots$

$$E(e^{im\phi}) = E(e^{-im\phi}). \quad (6)$$

Hence, letting $n = 1$, $m = \pm 1$ in Eq. (4) and subtracting, we obtain

$$E(y^2\cos\phi) = cE(\sin^2\phi). \quad (7)$$

Now let $n = 2$, $m = 0$; this yields

$$2E(ay^2 + by^2\cos\phi) = DE(1) = D. \quad (8)$$

Combining Eqs. (7) and (8) produces

$$aE(y^2) + bcE(\sin^2\phi) = \frac{D}{2}, \quad (9)$$

which is equivalent to Eq. (2) since $E(y) = E(\sin\phi) = 0$.

Eq. (2) is important from the design standpoint, since it determines the variance of either $\sin\phi$ or $\dot{\phi}$ when the other is known. In addition, Eq. (2) implies the upper bounds

$$\mathrm{Var}(\dot{\phi}) \leq \frac{D}{2a},$$

$$\mathrm{Var}(\sin\phi) \leq \frac{D}{2abc}.$$

D. The Effects of Radio Frequency (RF) Timing Noise in Two-Way Communication Systems

W. C. Lindsey

The subject of coherent two-way communication systems is an area which is least understood both by the communication theorist and in the laboratory. Although the basic form of a two-way coherent communication system has been established and operated, emphasis must now be placed on specifying system performance, specifying optimum design trends, and seeking means of improving performance by extending present-day techniques.

In the past, the relative success with which a digital communication system performs (one-way or two-way) in the presence of noise has been prescribed on a theoretical basis in terms of a probability of error versus signal-to-noise ratio (SNR) characteristic. Associated with any communication system, which has been designed on the basis of this theoretical characteristic, is an experimentally observable operating characteristic which relates the error probability, P_E, in the transmission and reception of one binary digit to some appropriate signal-to-noise ratio (SNR) existing in the receiver. In practice, this observable operating characteristic is inferior to the theoretical characteristic and consistently takes on a shape which is offset from the theoretical (ideal) curve derived by the theoretician. This offset usually reflects the "goodness" of the particular design. Thus, the design engineer seeks means and ways of explaining this offset and ways of adjusting his design so as to narrow the offset distance.

Quite frequently, as is the case in space communications or situations where multipath is a problem, a large part of this offset is directly traceable to the failure of the system to maintain sufficiently accurate reference models of the transmitted waveforms. In particular, the inaccuracies or uncertainties in stored reference models are primarily due to a phenomenon or disturbance which may be referred to as timing noise. In practice, this means that a filter which is designed to operate as a matched filter must operate (due to timing noise) as a "randomly" mismatched filter.

This note is the first in a sequel which reports on the effects of timing noise in coherent two-way communication systems. By timing noise we mean a random disturbance which introduces uncertainties in timing, e.g., the instantaneous phase of the RF carrier or subcarrier, matched filter readout instances, etc.

In two-way coherent communication systems, there are (at least) four principal sources of timing noise which affect the number of errors present in the recovered data. These are:

(1) Phase jitter on the RF carrier due to the additive noise on the up-link.

(2) Phase jitter on the RF carrier reference due to noise on the down-link.

(3) Phase jitter on the subcarrier due to additive noise on the down-link.

(4) Readout jitter on the bit sync signal due to additive noise on the down-link.

There are probably others which produce higher-order effects.

In this note we shall consider noise sources (1) and (2), and in a subsequent article treat sources (3) and (4) and the combined effects due to all four sources.

1. Basic Model

A two-way coherent communication link (Fig. 2) is one in which an RF carrier (possibly modulated) is transmitted to the vehicle, coherently detected, filtered, and retransmitted (after appropriate modulation) to the ground receiver. The situation may be explained in the following manner: The ground transmitter (Fig. 2) emits

$$\zeta(t) = 2^{1/2} A_1 \sin[\omega_0 t + \theta_m m(t)], \tag{1}$$

and the vehicle receiver observes the doppler-shifted, noise-corrupted version

$$\Psi(t) = 2^{1/2} A_1 \sin[\omega_1 t + \theta_m m(t) + \theta_1] + n_1(t), \tag{2}$$

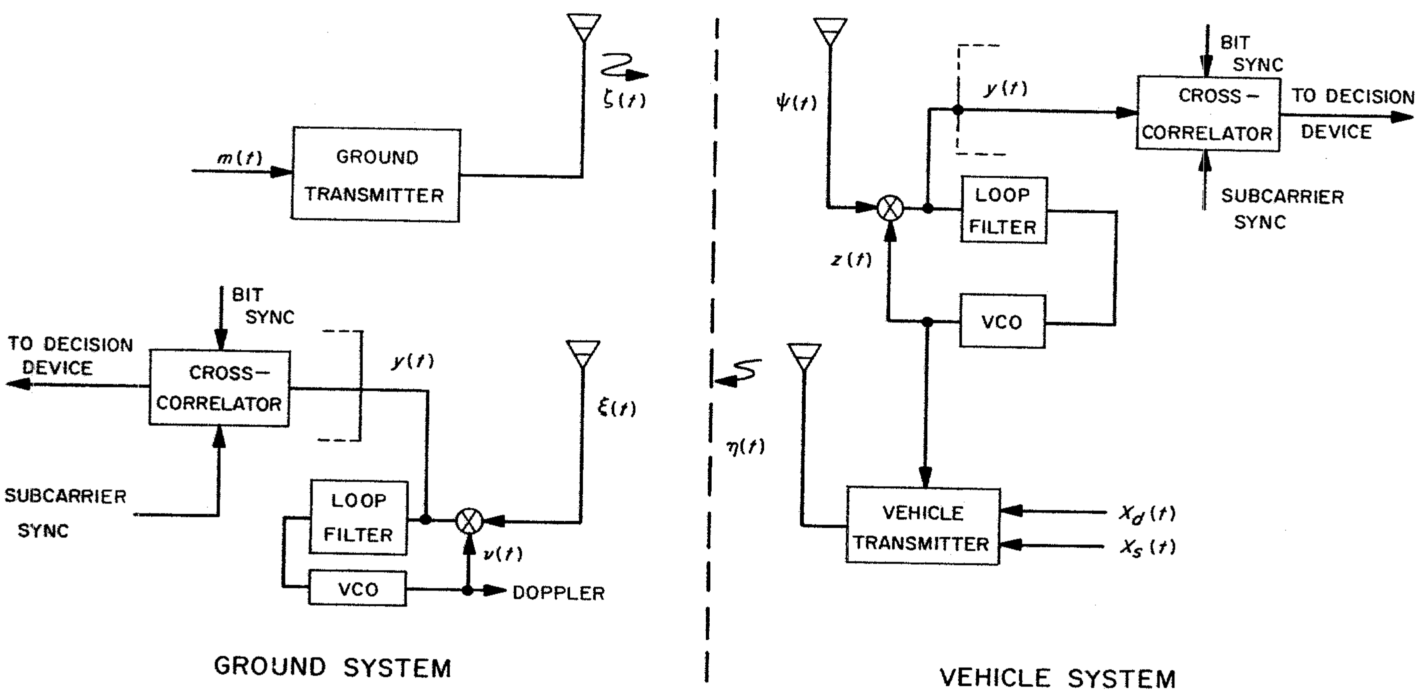

Fig. 2. Two-way communication link

where $n_1(t)$ is a narrow-band white Gaussian noise process which possesses a single-sided spectral density of N_{01} w/cps. The vehicle tracks (among other things) the carrier component in $\Psi(t)$ and provides the vehicle transmitter with the carrier component. Consequently, the vehicle transmitter emits

$$\eta(t) = 2^{1/2} A_2 \sin(\omega_1 t + \theta_d X_d(t) + \theta_s X_s(t) + \hat{\theta}_1), \tag{3}$$

where $X_s(t)$ is the sync subcarrier used for deriving timing information which is needed on the ground for operating the cross-correlator and $X_d(t)$ is the data signal. In some systems, $X_s(t)$ is a PN sequence whose fundamental frequency is commensurate with that of the data bits. For phase modulation (which we are considering here), $X_d(t)$ and $X_s(t)$ are unit square waves (± 1). At the ground receiver, one observes the doppler-shifted, noise-corrupted waveform

$$\begin{aligned} \xi(t) = {} & [2^{1/2} A_2 \cos\theta_d \cos\theta_s] \sin(\omega_2 t + \hat{\theta}_1 + \theta_2) \\ & + [2^{1/2} A_2 \sin\theta_d \cos\theta_s] X_d(t) \cos(\omega_2 t + \hat{\theta}_1 + \theta_2) \\ & + [2^{1/2} A_2 \sin\theta_s \cos\theta_d] X_s(t) \cos(\omega_2 t + \hat{\theta}_1 + \theta_2) \\ & + n_2(t), \end{aligned} \tag{4}$$

where we have used a simple trigonometric expansion and neglected cross-product terms which produce third- or higher-order effects. The additive disturbance $n_2(t)$ is a narrow-band white Gaussian noise process. The power in the received carrier component, the data component, and the sync subcarrier component is, respectively,

$$\begin{aligned} P_c &= A_2^2 \cos^2\theta_d \cos^2\theta_s \\ P_d &= A_2^2 \sin^2\theta_d \cos^2\theta_s \\ P_s &= A_2^2 \sin^2\theta_s \cos^2\theta_d. \end{aligned} \tag{5}$$

Thus, the cross-modulation loss is given by

$$P_l = A_2^2 \sin^2\theta_d \sin^2\theta_s, \tag{6}$$

since the total power P must sum to A_2^2; i.e.,

$$P = A_2^2 = P_l + P_c + P_d + P_s. \tag{7}$$

The ground receiver tracks the carrier component in $\xi(t)$, which provides the receiver with the estimate $\nu(t) = 2^{1/2}\cos(\omega_2 t + \hat{\theta}_2)$. Correspondingly, this component is used to coherently demodulate $\xi(t)$. The quantity $\hat{\theta}_2$ is the phase-locked loop estimate of $\hat{\theta}_1 + \theta_2$ and, as a result of this two-way tracking and demodulation process, there will be degradation in system performance due to the RF phase-jitter, $\hat{\theta}_2$. If, in fact, $\hat{\theta}_2 = \hat{\theta}_1 + \theta_2$ this jitter is zero, and one would observe no degradation in system performance due to the RF link. (Most systems have been designed on this false assumption.)

If we multiply the observed data $\xi(t)$ with the phase-locked loop estimate $\nu(t)$ and neglect the double frequency terms, we have

$$\begin{aligned} y(t) = {} & A_2 \sin\theta_d \cos\theta_s X_d(t) \cos[\hat{\theta}_2 - (\hat{\theta}_1 + \theta_2)] \\ & + A_2 \sin\theta_s \cos\theta_d X_s(t) \cos[\hat{\theta}_2 - (\hat{\theta}_1 + \theta_2)] + n(t), \end{aligned} \tag{8}$$

where $\theta_{RF} = \hat{\theta}_2 - (\hat{\theta}_1 + \theta_2)$ is the RF phase jitter (phase error, timing noise) and $n(t)$ is a narrow-band white Gaussian noise process possessing the single-sided spectral density of N_{02} w/cps. We wish to investigate the effects which this timing-noise component and the up- and down-link additive noise components introduce into the system performance characteristic.

In order to carry out this investigation, we shall assume that the demodulator (cross-correlator) is perfectly matched; i.e., bit sync and subcarrier sync are known exactly. First, however, we point out that the statistics of the RF phase error may be shown[3] to be given by

$$p(\theta_{RF}) = \frac{I_0(|\alpha_1 + \alpha_2 \exp(j\theta_{RF})|)}{I_0(\alpha_1) I_0(\alpha_2)}; \qquad |\theta_{RF}| \leqq \pi \tag{9}$$

where $I_0(x)$ is the Bessel function of zero order and imaginary argument,

$$\alpha_1 = \frac{(A_1 \cos\theta_m)^2}{N_{01} B_{L1}}$$

is the signal-to-noise ratio in the vehicle tracking loop bandwidth B_{L1}, and

$$\alpha_2 = \frac{(A_2 \cos\theta_s \cos\theta_d)^2}{N_{02} B_{L2}}$$

is the signal-to-noise ratio in the ground receiver tracking bandwidth B_{L2}.

[3]"Two-Way Doppler and Phase Measurements in Communication Networks," by W. C. Lindsey (to be published).

Without presenting the complicated and tedious details of the derivation, it is possible to show that the probability that the demodulator (cross-correlator) errs is given by

$$P_E(n) = \frac{1}{2}\left[1 - \left(\frac{8R_n}{\pi}\right)^{1/2} \sum_{k=0}^{\infty} \exp(-R_n)(-1)^k \varepsilon_k b_{2k+1}(n) \times (1-4k^2)^{-1} I_k(R_n)\right], \tag{10}$$

where

$$b_k(n) = \prod_{i=1}^{n} \frac{I_k(\alpha_i)}{I_0(\alpha_i)}; \qquad n = 1, 2.$$

$$\varepsilon_k = \begin{cases} 1; & \text{if } k = 0 \\ 2; & \text{if } k > 0 \end{cases}$$

$$R_n = S_n T_n / N_{0n}$$

or

$$R_1 = (A_1 \cos\theta_m)^2 T_1/N_{01} = S_1 T_1/N_{01}$$
$$= \text{data signal-to-noise ratio in vehicle}$$
$$R_2 = (A_2 \sin\theta_d \cos\theta_s)^2 T_2/N_{02} = S_2 T_2/N_{02}$$
$$= \text{data signal-to-noise ratio on ground}$$

If $n = 1$ in Eq. (10), we have the performance of the demodulator in the vehicle, while letting $n = 2$ yields the probability that the demodulator will make an error on the ground. If we fix the signal-to-noise ratios in the tracking loops and let R_n approach infinity, it is easy to show that the system is plagued with an irreducible error probability. This irreducible error rate is given, as a function of the signal-to-noise ratios in the tracking loops, by

$$P_E(n) = \frac{1}{2}\left[1 - \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\varepsilon_k b_{2k+1}(n)}{1-4k^2}\right] \tag{11}$$

A convenient check on the result is to let $\alpha_1 = \alpha_2$ approach infinity in Eq. (10). This corresponds to zero RF phase error, i.e., perfect measurement, and we have (Ref. 6)

$$P_E(n) = \frac{1}{2\pi}\int_{(2R_n)^{1/2}}^{\infty} \exp(-y^2/2)\,dy$$

which checks with previous results. For various values of R_n, Eq. (2) is plotted in Fig. 3 for the case where $\alpha_1 = \alpha_2 = \alpha$ and for $n = 1$ and 2. The dashed curves ($n = 2$) represent the error probability versus the parameter R_n when both the up-link and down-link are equally

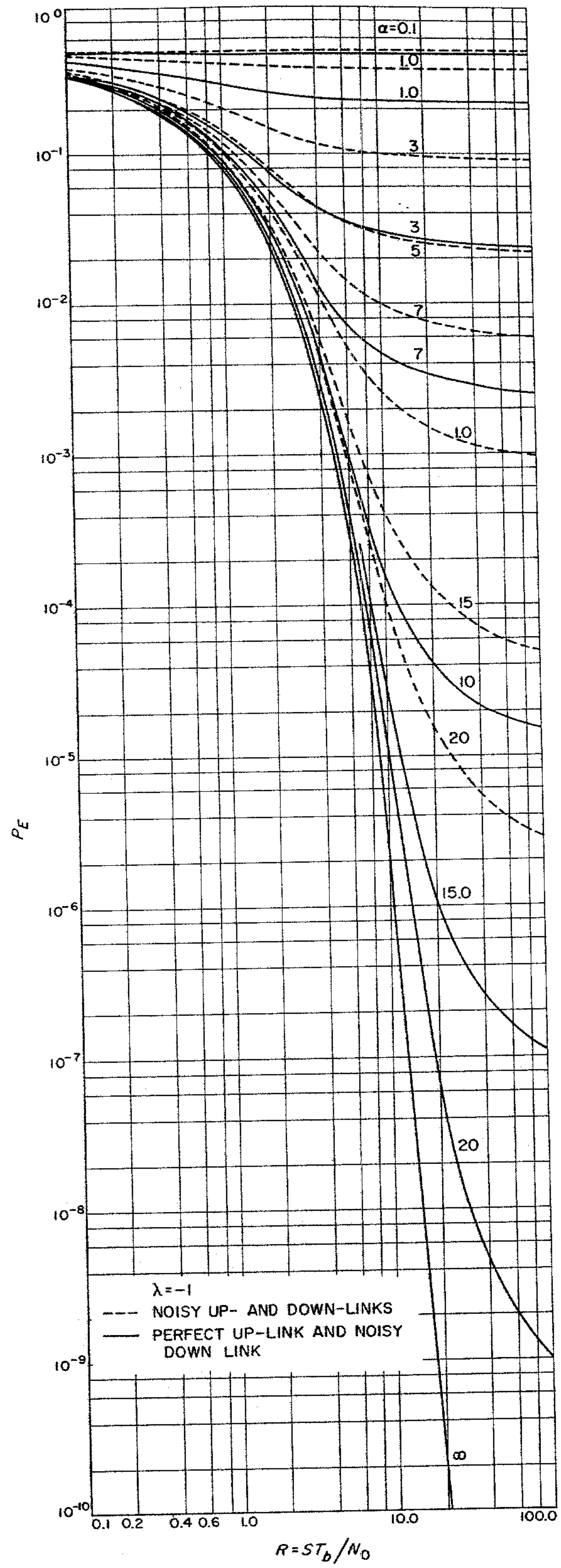

Fig. 3. Bit-error probability versus signal-to-noise ratio

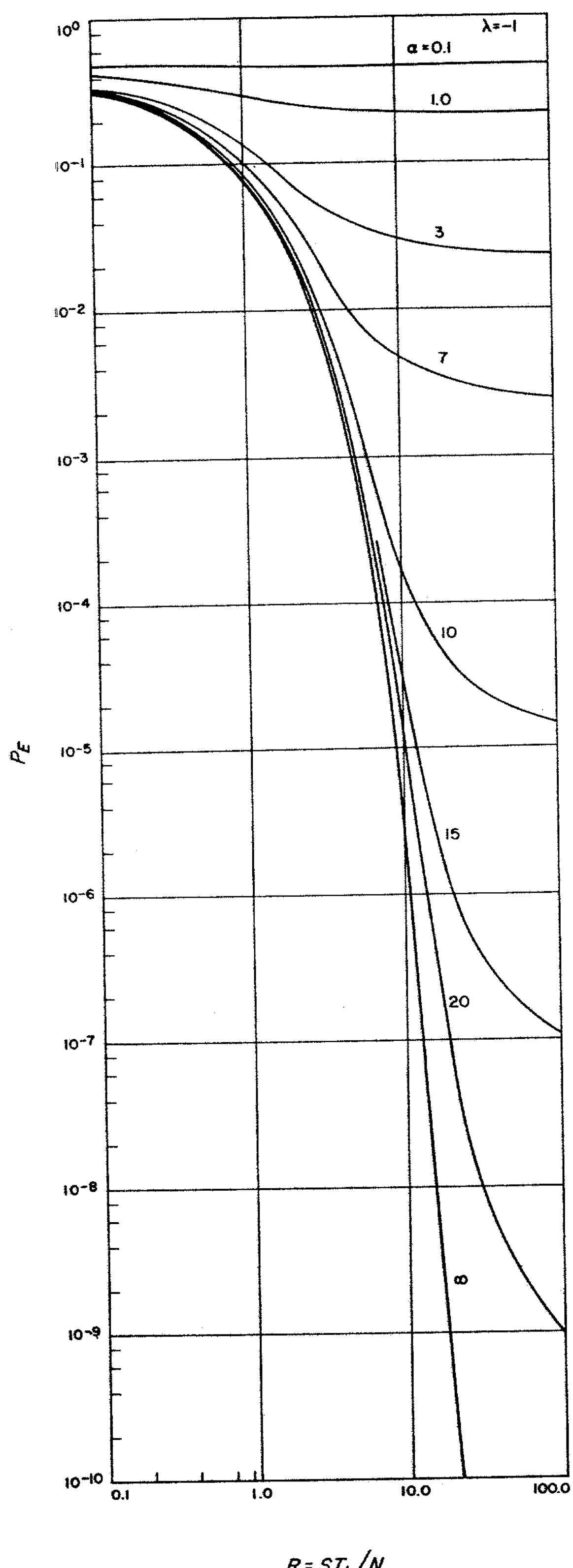

Fig. 4. Bit-error probability versus signal-to-noise ratio

noisy. The solid curves indicate the performance of the ground receiver when the up-link is non-noisy. Fig. 4 illustrates the performance of the vehicle receiver. Notice how the up-link timing noise affects the performance of the ground receiver and how the timing noise on the up- and down-links degrades over-all performance.

As an example of the use of these results, consider the situation where a two-way system has been designed to operate on the basis of zero RF-timing error and the $P_E(2) = 10^{-3}$. Suppose further that $\alpha_1 = \alpha_2 = \alpha = 15$. Entering Fig. 3 and using these values, we find that, to maintain an error rate of 10^{-3}, the signal-to-noise ratio in the data channel would have to be increased by approximately 2.4 db over what would be necessary if system timing references were perfect. This is a significant loss when one considers the cost per db of increasing the system signal-to-noise ratio. For one-way operation, the loss is approximately 0.6 db.

From a practical standpoint, it would be interesting to optimize system performance by varying the ground system or vehicle system modulation index so as to minimize $P_E(n)$ at both link ends. Further, it would be interesting to check these results with those obtained from experiment. Also of future interest is to determine optimum receiver structures, to extend the present analysis to "n-step" networks, to consider the effects of timing noise sources (3) and (4) and to consider the combined effects of all four sources.[4]

E. On the *A Priori* Information in Multi-Stage Estimation Problems

T. Nishimura

The optimal filter which has been introduced by Kalman (Refs. 7, 8) into the field of system theory yields the minimum variance estimate of states of linear systems, which are contaminated by white Gaussian noises, when a set of sequential observations is carried out. The basic feature of this filter is that the estimate of states is updated by a sequence of observations so as to minimize its

[4]Preliminary results which pertain to timing noise sources (3) and (4) are given in "The Detection of PSK Signals Using a Noisy Phase Reference," by W. C. Lindsey, *Proceedings of the National Telemetering Conference*, First Edition, pp. 50–53, April 1965.