

Q must be small compared to M . Note that if

$$\beta = \frac{S}{N_0 R} \log_e 2 > 2,$$

the upper and lower bounds on the error probability can be made arbitrarily small as k and Q are increased.

B. Another Look at the Optimum Design of Tracking Loops

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In 1955, Jaffe and Rehtin (Ref. 5) published the first sophisticated attempt at characterizing the optimum design of phase-locked loops. In the course of their work, they used an example which specified the transfer function of a loop best able to follow a frequency-step input insofar as minimizing transient error and phase noise are concerned. For simplicity, they assumed that the initial phase error was zero; the resulting filter function was one with one real zero and two complex poles, at a damping factor $\zeta = 0.707$, regardless of the initial frequency offset. The example was meant only to illustrate the optimization method, but since that time most systems have been designed using the parameters set by the example.

By using the same technique developed in the Jaffe-Rehtin paper, but assuming that the initial phase angle is random, a different result appears. Damping in the loop is always greater than $\zeta = 0.707$, and in all cases of practical interest, the system is *overdamped*. (Both poles lie on the negative real axis.)

Because the initial phase error is not generally known *a priori* (thus random), this latter design is one which seems to be of more practical use in most tracking applications.

1. Optimum Loops for Random Doppler Tracking

There are two sources of error during the initial acquisition of phase lock in a tracking receiver. First, there is a transient error as the system passes from its initial state to the steady-state tracking state. Second, there is phase jitter due to the presence of noise at the loop input. The technique developed by Jaffe and Rehtin was a Wiener

optimization of the linearized loop, but with a constraint on the total mean-square transient error. Following this technique, the optimum loop transfer function was found to be specified by the formula

$$H_{opt}(s) = \frac{1}{[\Psi(s)]^+} \left[\frac{\lambda^2 \mathcal{E} D(s) D(-s)}{[\Psi(s)]^-} \right]_{pr}, \quad (1)$$

where

$$\Psi(s) = \lambda^2 \mathcal{E} D(s) D(-s) + N_0/A^2$$

$D(s)$ = doppler-phase Laplace transform

λ^2 = Lagrange multiplier (to be evaluated)

A^2 = loop input carrier power

N_0 = double-sided noise spectral density

\mathcal{E} = expectation operator

$[\]^+$ = left half-plane "square-root" factorization operator

$[\]^-$ = right half-plane "square-root" factorization operator

$[\]_{pr}$ = $\mathcal{L}\mathcal{G}^{-1}$, the physical-realizability operator

The reader is referred to Ref. 3 or 5 for further explanation of the operators above and for the development of Eq. (1).

The optimization of interest is concerned with finding $H_{opt}(s)$ when the input doppler $d(t)$ has the form

$$d(t) = \theta_0 + \omega_0 t, \quad (2)$$

where θ_0 is a uniformly distributed phase angle, and where ω_0 is a random variable whose mean-square value is Ω_0^2 . The Laplace transform of $d(t)$ is

$$D(s) = \frac{\theta_0}{s} + \frac{\omega_0}{s^2}, \quad (3)$$

and hence the expected value of $D(s) D(-s)$ is

$$\mathcal{E} [D(s) D(-s)] = -\left(\frac{\pi^2}{3}\right) \frac{1}{s^2} + (\Omega_0^2) \left(\frac{1}{s^4}\right) \quad (4)$$

The first order of business is the factorization of $\Psi(s)$:

$$\begin{aligned} [\Psi(s)]^+ &= \frac{N_0^{1/2}}{As^2} \left[s^4 + \left(\frac{\pi^2 \lambda^2 A^2}{3N_0} \right) s^2 + \frac{\lambda^2 A^2 \Omega_0^2}{N_0} \right]^+ \\ &= \frac{N_0^{1/2}}{As^2} \left[s^2 + \left(\frac{A^2 \lambda^2 \pi^2}{3N_0} + \frac{2\lambda A \Omega_0}{N_0^{1/2}} \right)^{1/2} s + \frac{\lambda A \Omega_0}{N_0^{1/2}} \right]. \end{aligned} \quad (5)$$

It is convenient to define a quantity β^2

$$\beta^2 = \frac{\lambda A \Omega_0}{N_0^{1/2}} \quad (6)$$

The expression for $[\Psi(s)]^-$ is obtained by substituting $-s$ for s in $[\Psi(s)]^+$. Finally, evaluation of the $[\]_{pr}$ term yields the optimum transfer function

$$H_{opt}(s) = \frac{s \left[2\beta^2 + \frac{\pi^2 \beta^4}{3\Omega_0^2} \right]^{1/2} + \beta^2}{s^2 + \left[2\beta^2 + \frac{\pi^2 \beta^4}{3\Omega_0^2} \right]^{1/2} s + \beta^2} \quad (7)$$

The corresponding loop bandwidth can be computed by integration

$$\begin{aligned} 2B_L &= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} H_{opt}(s) H_{opt}(-s) ds \\ &= \frac{\beta \left(3 + \frac{\beta^2 \pi^2}{3\Omega_0^2} \right)}{2 \left(2 + \frac{\beta^2 \pi^2}{3\Omega_0^2} \right)^{1/2}} \end{aligned} \quad (8)$$

from which the Lagrange multiplier λ^2 can be evaluated in terms of B_L , Ω_0 , N_0 , and A^2 .

As we have indicated, these results are different than those contained in the Jaffe-Rechtin example. The filter they obtained, call it $H_{JR}(s)$, can be derived from the above by omitting the $\pi^2/3$ terms (i.e., by omitting those terms due the non-zero variance of θ_0):

$$\begin{aligned} H_{JR}(s) &= \frac{2^{1/2} \beta s + \beta^2}{s^2 + 2^{1/2} \beta s + \beta^2} \quad (9) \\ 2B_L &= \frac{3\beta}{2(2)^{1/2}} \end{aligned}$$

Note that $H_{JR}(s)$ always has a damping factor of $\zeta = 0.707$, whereas the damping factor of $H_{opt}(s)$ depends on several parameters. The optimum loop can thus be specified only when Ω_0 , A^2/N_0 , and B_L are given.

As a further consideration, one cannot expect a very good lock-in behavior when Ω_0 is so large that the incoming carrier frequency falls outside the loop pass-band. It seems very reasonable to optimize the loop when Ω_0 lies at the edge of the pass-band²

$$\Omega_0 = B_L \quad (10)$$

²This also corresponds to the case when a slow frequency sweep is being used, for then as Ω_0 comes within the pass-band of the loop, an optimum lock-up begins.

Under this assumption, Eq. (8) can be solved for β^2/Ω_0^2 numerically; the result is

$$\beta^2 = 1.67\Omega_0^2 = 1.67B_L^2 \quad (11)$$

By setting $\beta = 1.292B_L$ in Eq. (7) the Lagrange multiplier is removed, and the resulting optimum loop transfer function is

$$H_{opt}(s) = \frac{3.54B_L s + 1.67B_L^2}{s^2 + 3.54B_L s + 1.67B_L^2} \quad (12)$$

Specifically, $H_{opt}(s)$ has a zero at $-0.472B_L$, and poles at $-2.98B_L$ and $-0.56B_L$. Both of these poles lie on the negative real axis, indicating that the optimum system is overdamped.

The optimum loop filter is related to the over-all transfer function by

$$\begin{aligned} F_{opt}(s) &= \frac{s \left[2\beta^2 + \frac{\pi^2 \beta^4}{3\Omega_0^2} \right]^{1/2} + \beta^2}{AKs} \\ &= \frac{3.54B_L s + 1.67B_L^2}{AKs} \end{aligned} \quad (13)$$

in which K is the equivalent open-loop gain. It is usual to replace $F_{opt}(s)$ by a passive filter which degrades transient response only slightly if $AK \gg B_L^2$:

$$F_{opt}(s) \approx \frac{1 + (2.12/B_L) s}{1 + (0.611AK/B_L^2) s} \quad (14)$$

2. Evaluation of Transient Error

The total transient error is given by the expression

$$\begin{aligned} \epsilon^2 &= \\ &= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left(-\frac{\pi^2}{3s^2} + \frac{\Omega_0^2}{s^4} \right) [1 - H_{opt}(s)] [1 - H_{opt}(-s)] ds \end{aligned} \quad (15)$$

With the form of the optimum filter given in Eq. (7), the transient error is

$$\epsilon^2 = \frac{\Omega_0^2 \left[1 + \frac{\pi^2 \beta^2}{3\Omega_0^2} \right]}{2\beta^3 \left[2 + \frac{\pi^2 \beta^2}{3\Omega_0^2} \right]^{1/2}} \quad (16)$$

Under the constraint $B_L = \Omega_0$, the value of β^2 is $1.67B_L^2$, and the transient error is

$$\epsilon^2 = \frac{0.55}{B_L} \quad (17)$$

If $H_{JR}(s)$ had been used in Eq. (15) rather than $H_{opt}(s)$, the resulting transient error would have been

$$\epsilon_{JR}^2 = \frac{\Omega_0^2 \left[1 + \frac{\pi^2 \beta_{JR}^2}{3\Omega_0^2} \right]}{2(2)^{1/2} \beta_{JR}^3} \quad (18)$$

Given that $H_{JR}(s)$ and H_{opt} have equal bandwidths, from Eq. (9) the value of β_{JR} is $[4(2)^{1/2}/3] B_L$; the transient error thus arising by using the nonoptimal filter is

$$\epsilon_{JR}^2 = \frac{0.67}{B_L} \quad (19)$$

This figure indicates approximately a 1-db difference in transient performance.

3. Conclusions

In many spacecraft applications, it is necessary to design a phase-tracking system whose loop bandwidth is smaller than the initial frequency uncertainty interval, and to sweep the VCO slowly to acquire lock. This indicates that the phase-locked loop should be designed optimally to acquire and track once the carrier comes within the loop pass-band. The initial phase offset of the VCO is an unknown, completely random quantity, and the loop should be designed taking this factor into account. We have specified by Eqs. (12) and (13) what the optimum loop parameters are under these circumstances. The result is an *overdamped* system, as opposed to an underdamped ($\zeta = 0.707$) system predicted by the oversimplified Jaffe-Rechtin example.

The resulting decrease in transient error using the optimum loop as compared to *misusing* the Jaffe-Rechtin example, is only about 1 db, which may nevertheless be significant in the acquisition of threshold signals.

Whenever $AK \gg B_L^2$, the approximate optimum filter, given in Eq. (14) introduces a small amount of steady-state phase error, but this is generally small enough that the resulting performance of the loop is usually not degraded.

C. Some Moments Associated With Second-Order Phase-Locked Loops

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The differential equation governing the operation of a second-order phase-locked loop has the form

$$\ddot{\phi}(t) + [a + b \cos \phi(t)] \dot{\phi}(t) + c \sin \phi(t) = D(t), \quad (1)$$

where a, b, c are non-negative constants, ϕ is the "phase", and $D(t)$ is a certain Markovian noise process. (See *SPS 37-30* Vol. IV, pp. 262-268, for a general discussion of this problem.) We shall derive, from the Fokker-Planck equation associated with this equation, some equations which are satisfied by the expectations $E(\dot{\phi}^n e^{im\phi})$ (n, m integers). In particular, we shall show that the variances of $\dot{\phi}$ and $\sin \phi$ are connected by the simple relation

$$a \text{Var}(\dot{\phi}) + bc \text{Var}(\sin \phi) = \frac{D}{2}, \quad (2)$$

where D is the "mean second derivative" of the process $D(t)$, and depends only on $D(t)$.

1. Recursions for the Moments

The steady-state Fokker-Planck equation associated with Eq. (1) has the form

$$-y \frac{\partial p}{\partial \phi} + \frac{\partial}{\partial y} [(ay + by \cos \phi + c \sin \phi) p] + \frac{D}{2} \frac{\partial^2 p}{\partial y^2} = 0, \quad (3)$$

where we have written y in place of $\dot{\phi}$; $p = p(\phi, y)$ is the joint density function of the Markoff process (ϕ, y) ; $D = a$ a positive constant determined only by $D(t)$;

$$D = \lim_{\Delta t \rightarrow 0} \frac{E[\Delta D(t)]^2}{\Delta t}$$

It is evident from the nature of the problem that $p(\phi, y)$ is a periodic function in ϕ ; hence the moments $E(e^{im\phi})$ exist for all integers m . In the discussion that follows, we shall assume that the moments $E(|y|^n)$ exist and are finite for all positive integers n . If we multiply Eq. (3) by $y^n e^{im\phi}$, integrate by parts, and use this assumption and