

Chapter 3

The Continuum Model for Linear Arrays

All of the analysis presented so far has treated each oscillator as a discrete device with an injection port and an output port from which a signal emanates having a discrete phase value relative to a phase reference. For this reason, the mathematical model represented has been termed the discrete model. We emphasize that the discrete model encompasses the dynamic behavior of the oscillator array both nonlinear and, if desired, linearized. No new phenomena are added to this range of capability by means of the formulation to be discussed in the present chapter. However, it will be shown that, provided one is willing to linearize, the so called “continuum model” offers considerable advantage in terms of insight and applicability of familiar mathematical techniques. Although the continuum model is fundamentally approximate primarily because of the linearization, it nevertheless provides intuitive understanding of the behavior of coupled oscillator arrays with small inter-oscillator phase differences, an important special case in terms of practical application. Moreover, it provides a basis for understanding the impact of nonlinearity when the inter-oscillator phase differences increase beyond the limits of accurate linear approximation.

The continuum model in this context was suggested by Pogorzelski, et al. [38]. In essence we replace the index identifying the oscillators with a continuous variable such that, when the continuous variable takes on the value of the index for a given oscillator, a continuous function of that variable takes on the value of the phase of that oscillator. Thus, only the values of the function at integer values of its argument have physical meaning. The values between integer

values of the argument serve only to facilitate the formulation in terms of a differential equation.

3.1 The Linear Array without External Injection

To derive the continuum model of a simple linear array of oscillators coupled to nearest neighbors, we begin with Eq. (2.2-4) for the linearized discrete model *with zero coupling phase* and replace the discrete index i with a continuous variable, x .

$$\begin{aligned} \frac{d\varphi(x,t)}{dt} = \omega_0(x,t) - \omega_{ref} \\ + \Delta\omega_{lock} [\varphi(x + \Delta x, t) - 2\varphi(x, t) + \varphi(x - \Delta x, t)] \end{aligned} \quad (3.1-1)$$

where $\Delta x = 1$. Now treating $\varphi(x, t)$ as a continuous function of x , expanding each term in a Taylor series about x , and retaining terms up to second order in Δx , we obtain,

$$\frac{\partial\varphi(x,t)}{\partial t} = \omega_0(x,t) - \omega_{ref} + \Delta\omega_{lock} \frac{\partial^2\varphi(x,t)}{\partial x^2} \quad (3.1-2)$$

Finally, dividing by the locking range and using the normalized time variable, $\tau = \Delta\omega_{lock}t$, we have,

$$\frac{\partial\varphi(x,\tau)}{\partial\tau} = \Delta\Omega_{tune}(x,\tau) + \frac{\partial^2\varphi(x,\tau)}{\partial x^2} \quad (3.1-3)$$

This is the fundamental equation for the continuum model of a simple linear array of oscillators with nearest neighbor coupling and no external injection. It is the well-known diffusion equation. Laplace transformation with respect to time results in,

$$\frac{d^2\tilde{\varphi}(x,s)}{dx^2} - s\tilde{\varphi}(x,s) = -\Delta\tilde{\Omega}_{tune}(x,s) \quad (3.1-4)$$

a simple second-order linear differential equation for the transform of the phase distribution.

Suppose that the array is infinitely long and that one oscillator is step detuned at time zero by C locking ranges where C is less than two. Without loss of

generality, we may select the detuned oscillator to be the one at $x=0$. For this situation, Eq. (3.1-4) becomes,

$$\frac{d^2\tilde{\varphi}(x,s)}{dx^2} - s\tilde{\varphi}(x,s) = -\frac{C}{s}\delta(x) \quad (3.1-5)$$

As discussed in Ref. [38], it might be considered more correct to use, in place of the delta function, a square pulse one unit wide to represent the detuning. However, it is shown in Ref. [38] that the difference in the results is very small, and (in the spirit of the continuum model) the use of the delta function affords considerable convenience with minor impact on the results.

The differential equation given by Eq. (3.1-5) has an exact solution in closed form. It is,

$$\tilde{\varphi}(x,s) = \frac{C}{2s\sqrt{s}} e^{-|x|\sqrt{s}} \quad (3.1-6)$$

and the inverse Laplace transform is,

$$\varphi(x,\tau) = C\sqrt{\frac{\tau}{\pi}} e^{-x^2/(4\tau)} u(\tau) - \frac{C}{2}|x| \operatorname{erfc}\left(\frac{|x|}{2\sqrt{\tau}}\right) u(\tau) \quad (3.1-7)$$

Figure 3-1 shows a plot of this function over the range $-10 \leq x \leq 10$ from time zero to time equal to 250 inverse locking ranges for $C = 1$. Note that as time goes to infinity, the phase diverges as the square root of the time, never reaching a steady state. This may be viewed as a manifestation of the branch cut of Eq. (3.1-6) in the complex s plane. However, differentiating the phase with respect to time gives the simple expression for the frequency,

$$\frac{\omega(x,\tau) - \omega_{ref}}{\Delta\omega_{lock}} = \frac{C}{2} \sqrt{\frac{1}{\pi\tau}} e^{-x^2/(4\tau)} u(\tau) \quad (3.1-8)$$

and thus the frequency converges to the reference frequency at infinite time as one over the square root of the time. This function is plotted in Fig. 3-2 for C equal to unity.

Next, let us consider a finite length array over the range $-a - \frac{1}{2} \leq x \leq a + \frac{1}{2}$. For example, if $a = 10$ there will be 21 oscillators in the

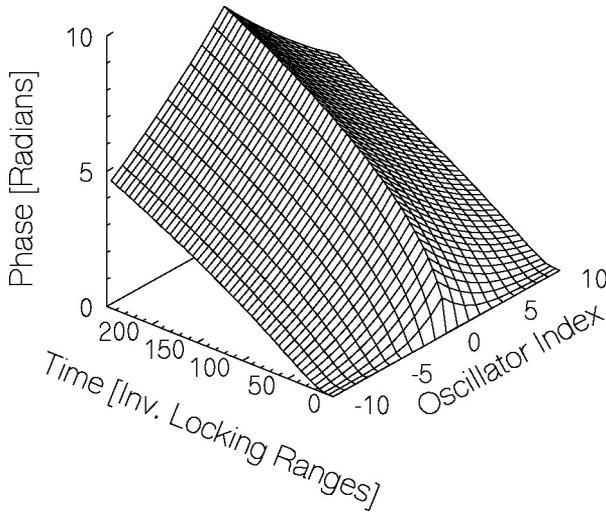


Fig. 3-1. Dynamic phase behavior of an infinite linear array.

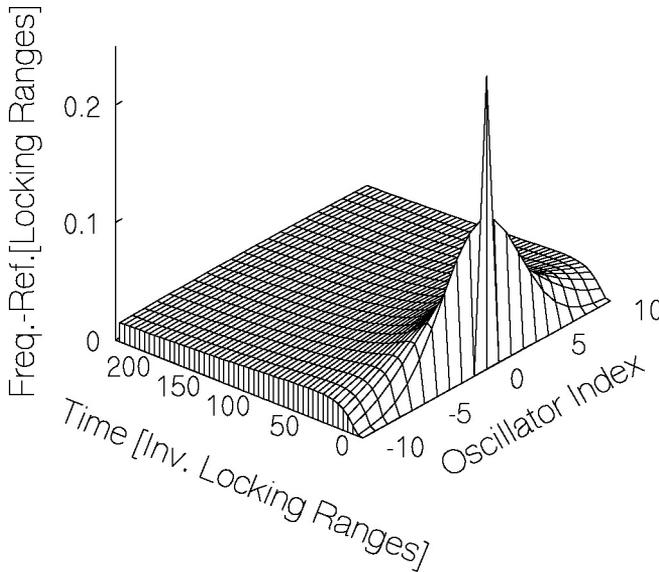


Fig. 3-2. Dynamic frequency behavior of an infinite linear array.

array and the overall length will be $2a+1$ or 21 unit cells. Now, in addition to using Eq. (3.1-4), we must determine the boundary conditions at the ends of the array in order to obtain the solution. These conditions can be easily obtained via an artifice outlined in Ref. [38]. That is, we imagine two additional fictitious oscillators added to the array, one at each end and coupled to the corresponding

end oscillator. These oscillators are dynamically tuned so that at all times their phase is maintained equal to the phase of the corresponding end oscillator of the true array. Under these conditions, as may be seen from Eqs. (1.4-1) and (1.4-2), there will be no mutual injection between the end oscillators and the fictitious ones. Thus, the fictitious ones may be removed without effect. However, since the phase of the end oscillator and the corresponding fictitious oscillator are always equal so that the phase difference is zero, and since in the continuum model this difference is represented by the derivative with respect to x , one may conclude that the appropriate boundary condition is that the derivative of the phase with respect to x must be zero; that is, a Neumann boundary condition. At this point, having both the differential equation Eq. (3.1-4) and the boundary conditions, we are in a position to treat the case of a finite length linear array via the continuum model. This will be accomplished using two alternative approaches described below both of which, of course, yield the same result.

Before proceeding on this course however, we note an interesting result obtainable directly from the differential equation and the boundary conditions. Suppose we integrate Eq. (3.1-3) over the length of the array.

$$\int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{\partial^2 \varphi(x, \tau)}{\partial x^2} dx - \frac{\partial}{\partial \tau} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \varphi(x, \tau) dx = - \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \Delta \Omega_{tune}(x, \tau) dx \quad (3.1-9)$$

The first term is zero by virtue of the Neumann boundary conditions at the array ends. Thus, we may write,

$$\frac{1}{2a+1} \frac{\partial}{\partial \tau} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \varphi(x, \tau) dx = \frac{1}{2a+1} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \Delta \Omega_{tune}(x, \tau) dx \quad (3.1-10)$$

or,

$$\frac{1}{2a+1} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{\partial \varphi(x, \tau)}{\partial t} dx = \frac{1}{2a+1} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \omega_{tune}(x, \tau) dx - \omega_{ref} \quad (3.1-11)$$

Now from Eq. (1.3-6), neglecting amplitude variation, we have that the instantaneous frequencies of the oscillators are given by,

$$\omega_{inst} = \omega_{ref} + \frac{\partial \varphi}{\partial t} \quad (3.1-12)$$

Substituting this into (3.1-11),

$$\frac{1}{2a+1} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \omega_{inst} dx = \frac{1}{2a+1} \int_{-a-\frac{1}{2}}^{a+\frac{1}{2}} \omega_{tune}(x, \tau) dx \quad (3.1-13)$$

That is, the average over the array of the instantaneous oscillator frequencies is equal to the average over the array of the oscillator tuning (or free running) frequencies. In steady state the instantaneous frequency is equal to the ensemble frequency. So, we can conclude that *the steady-state ensemble frequency of the array is the average of the oscillator tuning frequencies.* (Recall the assumption of zero coupling phase.)

We now set ourselves the problem of determining the phase dynamics of a finite linear array when one oscillator in the array is step detuned at time zero. The solution of this problem will be a Green's function permitting solution for an arbitrary distribution of detuning including the antisymmetrical detuning of the end oscillators for beam-steering as suggested by Liao and York [28]. The first approach will be to construct a solution as a superposition of a particular integral and two homogeneous solutions of the differential equation. The particular integral is known from the solution of the infinite array problem. It is essentially Eq. (3.1-6) generalized to accommodate detuning an arbitrary oscillator at $x = b$ instead of the one at $x = 0$. That is,

$$\tilde{\varphi}_p(x, s) = \frac{C}{2s\sqrt{s}} e^{-|x-b|\sqrt{s}} \quad (3.1-14)$$

Adding to this two independent homogeneous solutions with unknown coefficients, C_R and C_L , we postulate the desired solution in the form,

$$\tilde{\varphi}(x, s) = \frac{C}{2s\sqrt{s}} e^{-|x-b|\sqrt{s}} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} \quad (3.1-15)$$

The two unknown coefficients are now determined by applying the boundary conditions at the two ends of the array, $x = a + \frac{1}{2}$ and $x = -a - \frac{1}{2}$, resulting in the two simultaneous linear equations,

$$-C_R\sqrt{s}u + C_L\sqrt{s}\frac{1}{u} = -\frac{C}{2su}e^{-b\sqrt{s}} \quad (3.1-16)$$

$$-C_R\sqrt{s}\frac{1}{u} + C_L\sqrt{s}u = \frac{C}{2su}e^{b\sqrt{s}} \quad (3.1-17)$$

where,

$$u = e^{(a+\frac{1}{2})\sqrt{s}} \quad (3.1-18)$$

Solving Eqs. (3.1-16) and (3.1-17) simultaneously for C_R and C_L , we obtain,

$$C_R = \frac{-\frac{C}{2s\sqrt{s}}\left(\frac{1}{u^2}e^{b\sqrt{s}} + e^{-b\sqrt{s}}\right)}{\left(\frac{1}{u^2} - u^2\right)} \quad (3.1-19)$$

and,

$$C_L = \frac{-\frac{C}{2s\sqrt{s}}\left(\frac{1}{u^2}e^{-b\sqrt{s}} + e^{b\sqrt{s}}\right)}{\left(\frac{1}{u^2} - u^2\right)} \quad (3.1-20)$$

The solution given by Eq. (3.1-15) is then,

$$\begin{aligned} \tilde{\varphi}(x, s) = & \frac{C}{2s\sqrt{s}}e^{|x-b|\sqrt{s}} \\ & - \frac{\frac{C}{2s\sqrt{s}}\left(\frac{1}{u^2}e^{b\sqrt{s}} + e^{-b\sqrt{s}}\right)}{\left(\frac{1}{u^2} - u^2\right)}e^{-x\sqrt{s}} \\ & - \frac{\frac{C}{2s\sqrt{s}}\left(\frac{1}{u^2}e^{-b\sqrt{s}} + e^{b\sqrt{s}}\right)}{\left(\frac{1}{u^2} - u^2\right)}e^{x\sqrt{s}} \end{aligned} \quad (3.1-21)$$

which simplifies to,

$$\tilde{\varphi}(x, s) = \frac{C \cosh \left[(2a+1-|x-b|)\sqrt{s} \right] + C \cosh \left[(x+b)\sqrt{s} \right]}{s\sqrt{s} \sinh \left[(2a+1)\sqrt{s} \right]} \quad (3.1-22)$$

Note that, despite the presence of square roots of s , there are no branch cuts in the s plane because this function is even in the square root of s . Thus, the inverse Laplace transform can be computed purely via residue calculus. The poles, s_n , are located by,

$$s_n \sqrt{s_n} \sinh \left[(2a+1)\sqrt{s_n} \right] = 0 \quad (3.1-23)$$

Thus,

$$s_n = -\left(\frac{n\pi}{2a+1} \right)^2 = -\sigma_n \quad (3.1-24)$$

Except for the double pole at $s = 0$, the residues at these poles are,

$$residue_n = (-1)^n C \frac{\cosh \left[(2a+1-|x-b|)\sqrt{s_n} \right] + \cosh \left[(x+b)\sqrt{s_n} \right]}{s_n(2a+1)} \quad (3.1-25)$$

and the residue at the double pole is,

$$residue_0 = \frac{C}{2a+1} \quad (3.1-26)$$

The inverse Laplace transform is thus,

$$\begin{aligned} \varphi(x, \tau) = & \frac{C\tau}{2a+1} \\ & + \frac{C}{2a+1} \sum_{n=1}^{\infty} \frac{\cos\left[(x-b)\left(\frac{n\pi}{2a+1}\right)\right] + (-1)^n \cos\left[(x+b)\left(\frac{n\pi}{2a+1}\right)\right]}{\left(\frac{n\pi}{2a+1}\right)^2} \\ & \times \left(1 - e^{-\left(\frac{n\pi}{2a+1}\right)^2 \tau}\right) \end{aligned} \quad (3.1-27)$$

This may be rewritten in the form,

$$\begin{aligned} \varphi(x, \tau) = & \frac{C\tau}{2a+1} \\ & + \frac{C}{2a+1} \sum_{m=1}^{\infty} \frac{2 \cos\left[b\left(\frac{2m\pi}{2a+1}\right)\right] \cos\left[x\left(\frac{2m\pi}{2a+1}\right)\right]}{\left(\frac{2m\pi}{2a+1}\right)^2} \left(1 - e^{-\left(\frac{2m\pi}{2a+1}\right)^2 \tau}\right) \\ & + \frac{C}{2a+1} \sum_{n=0}^{\infty} \frac{2 \sin\left[b\left(\frac{(2n+1)\pi}{2a+1}\right)\right] \sin\left[x\left(\frac{(2n+1)\pi}{2a+1}\right)\right]}{\left(\frac{(2n+1)\pi}{2a+1}\right)^2} \left(1 - e^{-\left(\frac{(2n+1)\pi}{2a+1}\right)^2 \tau}\right) \end{aligned} \quad (3.1-28)$$

The overall time constant of the array dynamics is determined by the smallest eigenvalue. In general, this is given by the $n = 0$ term in Eq. (3.1-28); that is,

$$\sigma_0 = \left(\frac{\pi}{2a+1}\right)^2 \quad (3.1-29)$$

However, if the detuned oscillator happens to be the center one, the residues of the n series are zero and the smallest eigenvalue is the one for $m = 1$; that is,

$$\sigma_0 = \left(\frac{2\pi}{2a+1} \right)^2 \quad (3.1-30)$$

Thus, when the center oscillator is detuned, the array responds four times faster than if any other oscillator is detuned. (There is an error in Ref. [38] where this response is claimed to be only twice as fast.)

Recall now that from Eq. (3.1-13) the ensemble frequency of the array is the average of the tuning frequencies. When one oscillator out of the $2a+1$ oscillator array is detuned by C locking ranges, the ensemble frequency of the array measured in locking ranges will thus change by $C/(2a+1)$ locking ranges. This is manifest in the solution Eq. (3.1-28) as the linear time dependence of slope $C/(2a+1)$ as a function of the scaled time, τ . Aside from this linear term, from Eq. (3.1-28) we see that the steady-state phase distribution across the array is given by,

$$\begin{aligned} \varphi_{ss}(x) = & \frac{C}{2a+1} \sum_{n=1}^{\infty} \frac{2 \cos \left[b \left(\frac{2n\pi}{2a+1} \right) \right] \cos \left[x \left(\frac{2n\pi}{2a+1} \right) \right]}{\left(\frac{2n\pi}{2a+1} \right)^2} \\ & + \frac{C}{2a+1} \sum_{n=0}^{\infty} \frac{2 \sin \left[b \left(\frac{(2n+1)\pi}{2a+1} \right) \right] \sin \left[x \left(\frac{(2n+1)\pi}{2a+1} \right) \right]}{\left(\frac{(2n+1)\pi}{2a+1} \right)^2} \end{aligned} \quad (3.1-31)$$

a Fourier series which can be summed in closed form to yield the simple expression,

$$\varphi_{ss}(x) = \frac{C}{2(2a+1)} \left[x^2 + b^2 - (2a+1)|b-x| + \frac{1}{6}(2a+1)^2 \right] \quad (3.1-32)$$

This may be compared with the result from the discrete model where we approximated the eigenvalues and extended the sums to an infinite number of terms to arrive at the simple approximate result Eq. (2.3-9). Recall that in the linearized discrete model the eigenvalues repeat so, if the sums are continued to an infinite number of terms, a set of delta functions results. Here, in contrast,

the sums are in fact infinite and result in a smooth function passing through the correct value of oscillator phase as x passes through the corresponding index of that oscillator. Thus, the two results, discrete and continuum, are only equal at the oscillators and not in between.

As indicated in Ref. [38], because the inter-oscillator phase difference cannot exceed $\pi/2$, this steady-state result indicates that the detuning C is limited by,

$$C < \frac{\pi(2a+1)}{2(a+|b|)} \tag{3.1-33}$$

However, when operating near the limits of lock, this is not a very good approximation so it is suggested in [38] that the sine terms be approximated by defining an effective locking range, $\Delta\tilde{\omega}_{lock}$, as follows.

$$\Delta\omega_{lock} \sin(\Delta\phi) \approx \Delta\omega_{lock} \frac{\sin(\Delta\phi)}{\Delta\phi} \Delta\phi = \Delta\tilde{\omega}_{lock} \Delta\phi \tag{3.1-34}$$

For small phase differences the effective locking range will be nearly equal to the true locking range, but near the limits of lock, it will be $2/\pi$ times the true locking range. Thus, as pointed out in Ref. [38], though still approximate, the maximum detuning is more accurately given by,

$$\Delta\omega_{max} \approx \frac{(2a+1)}{(a+|b|)} \Delta\omega_{lock} \tag{3.1-35}$$

Let us now return to the problem of determining the phase dynamics of a finite linear array when one oscillator in the array is step detuned at time zero and solve it via an alternative approach. We wish to solve Eq. (3.1-5) subject to Neumann boundary conditions at the array ends. Following Pogorzelski, et al. [38] in this alternate approach we first determine the eigenfunctions and eigenvalues defined by,

$$\frac{d^2 w_\ell}{dx^2} = \lambda_\ell w_\ell \tag{3.1-36}$$

such that,

$$\left. \frac{dw_n}{dx} \right|_{x=a+\frac{1}{2}} = 0 \tag{3.1-37}$$

and

$$\left. \frac{dw_n}{dx} \right|_{x=-a-\frac{1}{2}} = 0 \quad (3.1-38)$$

Clearly, the appropriately normalized eigenfunctions are,

$$u_m = \frac{\sqrt{2} \cosh(x\sqrt{\lambda_m})}{\sqrt{2a+1}} \quad (3.1-39)$$

and

$$v_n = \frac{\sqrt{2} \sinh(x\sqrt{\lambda_n})}{i\sqrt{2a+1}} \quad (3.1-40)$$

and the eigenvalues are given by,

$$\sinh \left[\sqrt{\lambda_m} \left(a + \frac{1}{2} \right) \right] = 0 \quad (3.1-41)$$

and

$$\cosh \left[\sqrt{\lambda_n} \left(a + \frac{1}{2} \right) \right] = 0 \quad (3.1-42)$$

Thus the explicit eigenvalues are,

$$\lambda_m = - \left(\frac{2m\pi}{2a+1} \right)^2 \quad (3.1-43)$$

and

$$\lambda_n = - \left(\frac{(2n+1)\pi}{2a+1} \right)^2 \quad (3.1-44)$$

We now express the solution of Eq. (3.1-5) as a sum of these eigenfunctions. That is,

$$\tilde{\varphi} = \sum_{m=0}^{\infty} A_m u_m + \sum_{n=0}^{\infty} B_n v_n \quad (3.1-45)$$

Substituting this into Eq. (3.1-5) generalized to an arbitrary detuned oscillator at $x = b$ gives,

$$A_m \lambda_m u_m + B_n \lambda_n v_n - s A_m u_m + B_n v_n = -\frac{C}{s} \delta(x-b) \quad (3.1-46)$$

Now using the orthogonality of the eigenfunctions over the length of the array, we obtain,

$$A_m = -\frac{C u_m(b)}{s(\lambda_m - s)} \quad (3.1-47)$$

and

$$B_n = -\frac{C v_n(b)}{s(\lambda_n - s)} \quad (3.1-48)$$

The solution is then immediately written as,

$$\tilde{\varphi} = -C \sum_{m=0}^{\infty} \frac{u_m(b) u_m(x)}{s(\lambda_m - s)} + \sum_{n=0}^{\infty} \frac{v_n(b) v_n(x)}{s(\lambda_n - s)} \quad (3.1-49)$$

or, inserting the explicit expressions for the eigenfunctions,

$$\begin{aligned} \tilde{\varphi} = & -\frac{C}{s} \sum_{m=0}^{\infty} \frac{2 \cosh(b\sqrt{\lambda_m}) \cosh(x\sqrt{\lambda_m})}{(2a+1)(\lambda_m - s)} \\ & -\frac{C}{s} \sum_{n=0}^{\infty} \frac{2 \sinh(b\sqrt{\lambda_n}) \sinh(x\sqrt{\lambda_n})}{(2a+1)(\lambda_n - s)} \end{aligned} \quad (3.1-50)$$

Except for the zero eigenvalue term, $m = 0$, each term of these series has one simple pole at s equal to the corresponding eigenvalue. Thus, the inverse Laplace transform follows immediately as the sum of the residues at the pole in each term of the series,

$$\begin{aligned}
\varphi(x, \tau) = & \frac{C\tau}{2a+1} \\
& + \frac{C}{2a+1} \sum_{m=1}^{\infty} \frac{2 \cos \left[b \left(\frac{2m\pi}{2a+1} \right) \right] \cos \left[x \left(\frac{2m\pi}{2a+1} \right) \right]}{\left(\frac{2m\pi}{2a+1} \right)^2} \left(1 - e^{-\left(\frac{2m\pi}{2a+1} \right)^2 \tau} \right) \\
& + \frac{C}{2a+1} \sum_{n=0}^{\infty} \frac{2 \sin \left[b \left(\frac{(2n+1)\pi}{2a+1} \right) \right] \sin \left[x \left(\frac{(2n+1)\pi}{2a+1} \right) \right]}{\left(\frac{(2n+1)\pi}{2a+1} \right)^2} \left(1 - e^{-\left(\frac{(2n+1)\pi}{2a+1} \right)^2 \tau} \right)
\end{aligned} \tag{3.1-51}$$

which is, of course, identical to Eq. (3.1-28). For the case where $a = 10$ and $b = 5$, this solution is plotted as a function of time in Fig. 3-3. Note that the shape of the distribution at late times is very much like the corresponding steady-state solution shown in Fig. 2-3. Being the solution for a delta function source on the right side of the differential equation, this is the Green's function for the problem and as such it can be used to obtain solutions for arbitrary detuning distributions.

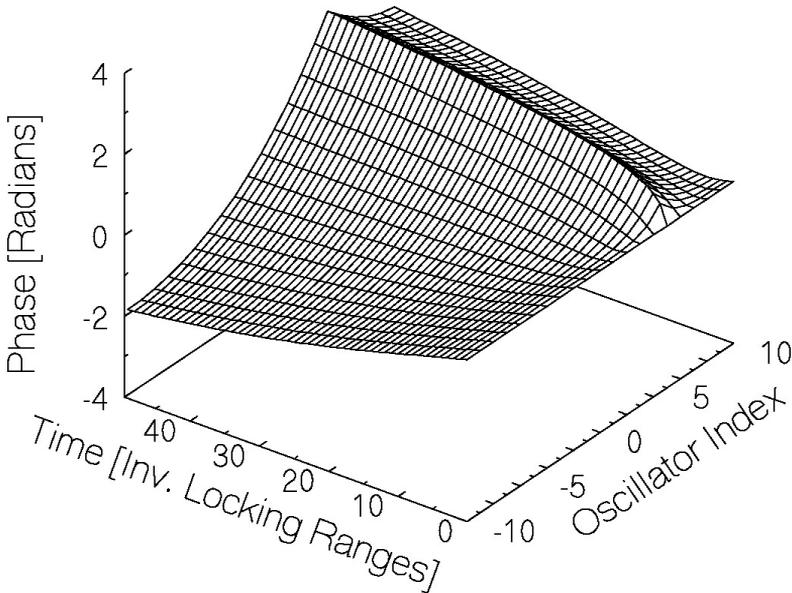


Fig. 3-3. Linear array phase distribution under step detuning of the oscillator at $x = 5$.

To summarize, we have described two methods of solving the continuum-model partial-differential equation for the dynamic behavior of the phase across a linear array of mutually injection locked oscillators. Both methods entailed Laplace transformation with respect to the scaled time. The first method was a direct solution of the resulting second-order ordinary differential equation by postulating a solution as a superposition of a particular integral and two homogeneous solutions with unknown amplitude coefficients. The coefficients were determined by the Neumann boundary conditions at the array ends. The inverse Laplace transform was obtained as a sum of the residues of at the poles of the transform. In the second method, the Laplace transformed equation was solved by postulating a solution as a sum of eigenfunctions of the second order differential operator each satisfying the Neumann boundary conditions at the array ends. Recognizing this to be a self-adjoint boundary value problem of Sturm-Liouville type, it should not be surprising that the solution for the desired Green's function can be written as a sum of these eigenfunctions. Conveniently, each term of the sum, except the one corresponding to the zero eigenvalue, has one simple pole so that the inverse Laplace transform is immediately obtainable as a sum of the corresponding residues, one for each term of the eigenfunction series.

3.2 The Linear Array with External Injection

Thus far, the continuum model has been applied to arrays in which the phase control is accomplished by detuning one of the oscillators. The beam-steering method proposed by Stephan [1] requires that two or more array oscillators be injected with an externally derived signal. Thus, to accommodate this, it is necessary to generalize the continuum model along the lines followed in Section 2.5. Following Pogorzelski, et al. [39], we begin with Eq. (2.5-2) rewritten in terms of the continuous variable, x , and the scaled time, τ , as,

$$\begin{aligned} \frac{d\varphi(x, \tau)}{d\tau} = & \frac{\omega_{0i} - \omega_{ref}}{\Delta\omega_{lock}} + (\varphi(x + \Delta x, \tau) - 2\varphi(x, \tau) + \varphi(x - \Delta x, \tau)) \\ & - \delta(x - p) \frac{\Delta\omega_{lock, p, inj}}{\Delta\omega_{lock}} (\varphi(x, \tau) - \varphi_{inj}(\tau)) \end{aligned} \quad (3.2-1)$$

Now we define,

$$V(x) = \delta(x - p) \frac{\Delta\omega_{lock, p, inj}}{\Delta\omega_{lock}} \quad (3.2-2)$$

and expand in a Taylor series about x keeping terms up to second order in Δx so that Eq. (3.2-1) becomes,

$$\frac{\partial^2 \varphi}{\partial x^2} - V(x)\varphi - \frac{d\varphi}{d\tau} = -\Delta\Omega_{tune} - V(x)\varphi_{inj}(\tau) \quad (3.2-3)$$

Here the spatial distribution of the external injection signals is given by $V(x)$ while the temporal dependence is given by $\varphi_{inj}(\tau)$ so we have implicitly assumed that these dependences are separable; that is, all of the injection signal phases have the same time dependence. While this is a convenient simplification, it is not essential in that one could include more than one such injection term in the equation and obtain a solution albeit somewhat more complicated than the one presented here. Equation (3.2-3) is the generalization of Eq. (3.1-3) required to accommodate external injection for our purposes and we will use it to study the phase dynamics of such an externally injected array.

Suppose we consider an infinitely long linear array wherein all of the oscillators are tuned to the ensemble or reference frequency and the oscillator at $x = b$ is externally injection locked to an oscillator of strength C with C_0 radian step time dependence of its phase. Our generalized differential equation then becomes,

$$\frac{\partial^2 \varphi}{\partial x^2} - C\delta(x-b)\varphi - \frac{d\varphi}{d\tau} = -\Delta\Omega_{tune} - CC_0\delta(x-b)u(\tau) \quad (3.2-4)$$

where,

$$C = \frac{\Delta\omega_{lock,p,inj}}{\Delta\omega_{lock}} \quad (3.2-5)$$

Laplace transformation with respect to the scaled time results in,

$$\frac{\partial^2 \tilde{\varphi}}{\partial x^2} - C\delta(x-b)\tilde{\varphi} - s\tilde{\varphi} = -\frac{C_0}{s}C\delta(x-b) \quad (3.2-6)$$

We now define,

$$\tilde{\varphi}_1 = \tilde{\varphi} - \frac{C_0}{s} \quad (3.2-7)$$

so that Eq. (3.2-6) becomes,

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial x^2} - C\delta(x-b)\tilde{\varphi}_1 - s\tilde{\varphi}_1 = C_0 \quad (3.2-8)$$

The particular integral of this equation is,

$$\tilde{\varphi}_{1p} = -\frac{C_0}{s} \quad (3.2-9)$$

We postulate a homogeneous solution of the form,

$$\tilde{\varphi}_{1h} = C_1 e^{-\sqrt{s}|x-b|} \quad (3.2-10)$$

so that our proposed solution is,

$$\tilde{\varphi}_1 = -\frac{C_0}{s} + C_1 e^{-\sqrt{s}|x-b|} \quad (3.2-11)$$

Now integrating Eq. (3.2-8) across the delta function at $x = b$, we find that,

$$\left. \frac{d\tilde{\varphi}_1}{dx} \right|_{x=b^-}^{x=b^+} = C\tilde{\varphi}_1(b) \quad (3.2-12)$$

Imposing this condition on the solution given by Eq. (3.2-11), we obtain,

$$C_1 = \frac{C_0 C}{s(2\sqrt{s} + C)} \quad (3.2-13)$$

Substituting this into the solution given by Eq. (3.2-11) gives,

$$\tilde{\varphi}_1 = \frac{C_0}{s} \left[\frac{C}{2\sqrt{s} + C} e^{-\sqrt{s}|x-b|} - 1 \right] \quad (3.2-14)$$

and from Eq. (3.2-7),

$$\tilde{\varphi}(x, s) = \frac{C_0 C}{s(2\sqrt{s} + C)} e^{-\sqrt{s}|x-b|} \quad (3.2-15)$$

Finally, the inverse Laplace transform of Eq. (3.2-15) is,

$$\begin{aligned} \varphi(x, \tau) = C_0 \left[\operatorname{erfc} \left(\frac{|x-b|}{2\sqrt{\tau}} \right) \right. \\ \left. - e^{C|x-b|/2} e^{C^2\tau/4} \operatorname{erfc} \left(C \frac{\sqrt{\tau}}{2} + \frac{|x-b|}{2\sqrt{\tau}} \right) \right] u(\tau) \end{aligned} \quad (3.2-16)$$

(See Ref. [37] equation 29.3.89.) This is the phase distribution across the infinite array as a function of time. It is zero at time zero and smoothly evolves to a final value of C_0 at infinite time as shown in Fig. 3-4 for $C_0 = 1$ radian and $C = 1$. Note that the injection frequency as well as the initial and final ensemble frequencies are all the same. Because it is the solution for injection at a single point in the array, you might think that it is a Green's function that can be used to construct solutions for arrays injected at multiple points. However, as we shall see in Section 3.4 when we discuss Stephan's beam-steering scheme [1] involving two injection points, this is not the case because the form of differential equation itself differs from Eq. (3.2-6) when there are multiple injection points.

The corresponding problem where the injected *frequency* is step shifted by C_0 locking ranges at time zero was treated by Pogorzelski, et al. [39]. In that case the array oscillator frequencies evolve from the ensemble frequency at time zero to the injection frequency at infinite time.

Next, we consider an array of finite length, $2a + 1$, in which all of the oscillators are tuned to the same frequency, taken to be the reference frequency and one of the oscillators, the one at $x = b$, is injected with an externally generated signal of strength C defined by Eq. (3.2-5) that is step phase shifted at time zero by C_0 radians. Equation (3.2-6) applies, but this time we wish to solve it subject to Neumann boundary conditions at the array ends. Here again we have a choice of two methods of solution. Let us begin by postulating the solution in the form of a particular integral plus two complementary functions that are solutions of the homogeneous equation. That is, using Eqs. (3.2-7), (3.2-9), and (3.2-10) we have,

$$\tilde{\varphi}_1 = C_b e^{-\sqrt{s}|x-b|} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} - \frac{C_0}{s} \quad (3.2-17)$$

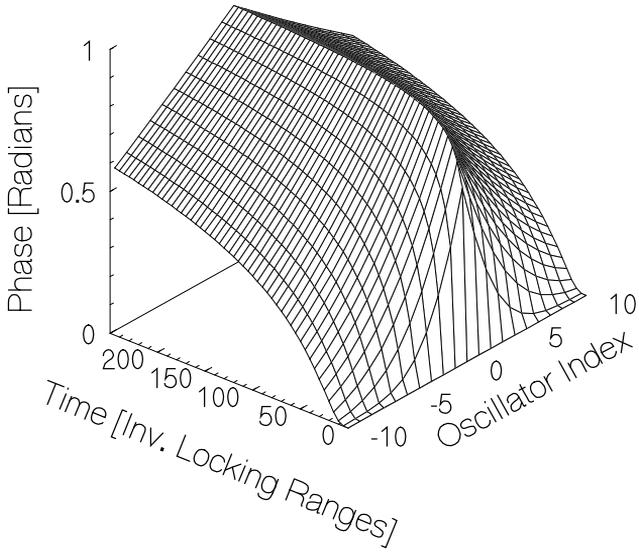


Fig. 3-4. Phase distribution versus time for an infinite linear array with one oscillator externally injected.

with the three conditions,

$$\left. \frac{d\tilde{\varphi}_1}{dx} \right|_{x=b^+} = C\tilde{\varphi}_1(b) \tag{3.2-18}$$

$$\left. \frac{d\tilde{\varphi}_1}{dx} \right|_{x=a} = 0 \tag{3.2-19}$$

$$\left. \frac{d\tilde{\varphi}_1}{dx} \right|_{x=-a} = 0 \tag{3.2-20}$$

Now, Eqs. (3.2-18), (3.2-19), and (3.2-20) can be used to determine the three constants, C_b , C_R , and C_L . Then, using Eq. (3.2-7), we get,

$$\tilde{\varphi}(x, s) = \frac{C_0}{2sD(s)} \left\{ C \cosh \left[(2a + 1 - |x - b|) \sqrt{s} \right] + C \cosh \left[(x + b) \sqrt{s} \right] \right\} \tag{3.2-21}$$

where,

$$D(s) = \sqrt{s} \sinh \left[(2a+1)\sqrt{s} \right] + C \cosh \left[\left(a + \frac{1}{2} + b \right) \sqrt{s} \right] \cosh \left[\left(a + \frac{1}{2} - b \right) \sqrt{s} \right] \quad (3.2-22)$$

Here again there are no branch cuts, and the inverse Laplace transform is expressible as a sum of residues at the poles; that is, the zeros of $D(s)$, all of which lie on the negative real axis of the s plane. Note that Eq. (3.2-22) is very reminiscent of Eq. (2.5-9) of the discrete model of this array. Comparing these two equations, we may ascertain that the continuum approximation is particularly accurate for small values of s when $\sqrt{s} \approx \sinh(\sqrt{s})$ which, of course, corresponds to late time. In fact, the pole closest to the origin of the s plane provides us with the time constant of the array which determines the late time behavior. Let us examine Eq. (3.2-22) to see if we can estimate the location of this pole.

In anticipation of the fact that the pole lies on the negative real axis, we define ξ so that,

$$\sqrt{s} = \sqrt{-\sigma} = i\sqrt{\sigma} = i\xi \quad (3.2-23)$$

Then,

$$D = -\xi \sin \left[(2a+1)\xi \right] + C \cos \left[\left(a + \frac{1}{2} + b \right) \xi \right] \cos \left[\left(a + \frac{1}{2} - b \right) \xi \right] \quad (3.2-24)$$

Setting D equal to zero, yields the transcendental equation,

$$2\xi \sin \left[\left(a + \frac{1}{2} \right) \xi \right] \cos \left[\left(a + \frac{1}{2} \right) \xi \right] = C \cos \left[\left(a + \frac{1}{2} + b \right) \xi \right] \cos \left[\left(a + \frac{1}{2} - b \right) \xi \right] \quad (3.2-25)$$

For small ξ , the solution occurs where the cosine functions are near zero and the sine function is near unity. Thus, we define a new variable,

$$\eta = \xi - \frac{\pi}{2a+1} \quad (3.2-26)$$

and write Eq. (3.2-25) in the form,

$$\begin{aligned}
2\left(\eta - \frac{\pi}{2a+1}\right) \sin\left[\left(a + \frac{1}{2}\right)\eta + \frac{\pi}{2}\right] \cos\left[\left(a + \frac{1}{2}\right)\eta + \frac{\pi}{2}\right] = \\
C \cos\left[\left(a + \frac{1}{2} + b\right)\eta + \frac{\pi}{2} + \frac{b\pi}{2a+1}\right] \\
\times \cos\left[\left(a + \frac{1}{2} - b\right)\eta + \frac{\pi}{2} - \frac{b\pi}{2a+1}\right]
\end{aligned} \tag{3.2-27}$$

or

$$\begin{aligned}
2\left(\eta - \frac{\pi}{2a+1}\right) \cos\left[\left(a + \frac{1}{2}\right)\eta\right] \sin\left[\left(a + \frac{1}{2}\right)\eta\right] = \\
-C \sin\left[\left(a + \frac{1}{2} + b\right)\eta + \frac{b\pi}{2a+1}\right] \\
\times \sin\left[\left(a + \frac{1}{2} - b\right)\eta - \frac{b\pi}{2a+1}\right]
\end{aligned} \tag{3.2-28}$$

Using the identity for the sine of a sum, we arrive at,

$$\begin{aligned}
2\left(\eta - \frac{\pi}{2a+1}\right) \cos\left[\left(a + \frac{1}{2}\right)\eta\right] \sin\left[\left(a + \frac{1}{2}\right)\eta\right] = \\
-C \left\{ \sin\left[\left(a + \frac{1}{2} + b\right)\eta\right] \cos\left(\frac{b\pi}{2a+1}\right) \right. \\
\left. + \cos\left[\left(a + \frac{1}{2} + b\right)\eta\right] \sin\left(\frac{b\pi}{2a+1}\right) \right\} \\
\times \left\{ \sin\left[\left(a + \frac{1}{2} - b\right)\eta\right] \cos\left(\frac{b\pi}{2a+1}\right) \right. \\
\left. - \cos\left[\left(a + \frac{1}{2} - b\right)\eta\right] \sin\left(\frac{b\pi}{2a+1}\right) \right\}
\end{aligned} \tag{3.2-29}$$

Near $\eta = 0$,

$$\begin{aligned}
\pi\eta = & \\
& C \left\{ \left[\left(a + \frac{1}{2} + b \right) \eta \right] \cos \left(\frac{b\pi}{2a+1} \right) \right. \\
& \quad \left. + \sin \left(\frac{b\pi}{2a+1} \right) \right\} \\
& \times \left\{ \left[\left(a + \frac{1}{2} - b \right) \eta \right] \cos \left(\frac{b\pi}{2a+1} \right) \right. \\
& \quad \left. - \sin \left(\frac{b\pi}{2a+1} \right) \right\}
\end{aligned} \tag{3.2-30}$$

which is a quadratic equation for η . That is,

$$\begin{aligned}
& \left[\left(a + \frac{1}{2} \right)^2 - b^2 \right] \eta^2 \\
& - \left[\frac{\pi}{C} \sec^2 \left(\frac{b\pi}{2a+1} \right) + 2b \tan \left(\frac{b\pi}{2a+1} \right) \right] \eta - \tan^2 \left(\frac{b\pi}{2a+1} \right) = 0
\end{aligned} \tag{3.2-31}$$

We can now look at two limiting cases. First, if C is small, the solution becomes that of the uninjected array, namely, $\eta = 0$. If, on the other hand, C is large,

$$\eta = \frac{\left[b \pm \left(a + \frac{1}{2} \right) \right]}{\left[\left(a + \frac{1}{2} \right)^2 - b^2 \right]} \tan \left(\frac{b\pi}{2a+1} \right) \tag{3.2-32}$$

and

$$\xi = \frac{\left[b \pm \left(a + \frac{1}{2} \right) \right]}{\left[\left(a + \frac{1}{2} \right)^2 - b^2 \right]} \tan \left(\frac{b\pi}{2a+1} \right) + \frac{\pi}{2a+1} \tag{3.2-33}$$

If b is small; that is, if the injection point is near the center of the array,

$$\xi \approx \frac{\left[b \pm \left(a + \frac{1}{2} \right) \right]}{\left[\left(a + \frac{1}{2} \right)^2 - b^2 \right]} \left(\frac{b\pi}{2a+1} \right) + \frac{\pi}{2a+1} \quad (3.2-34)$$

Choosing the sign in the numerator to obtain the solution nearest the origin of the s plane, we have,

$$\xi \approx \frac{\pi}{2a+1+2|b|} \quad (3.2-35)$$

Thus,

$$s_{\min} \approx - \left(\frac{\pi}{2(a+|b|+1)} \right)^2 \quad (3.2-36)$$

and that the late time behavior of the array goes as,

$$e^{- \left(\frac{\pi}{2(a+|b|+1)} \right)^2 \tau} \quad (3.2-37)$$

The formula given by Eq. (3.2-33) fails if b is at either end of the array because we have effectively divided by zero in the derivation. We can no longer assume that C is infinite. Retaining a finite value of C and rewriting the transcendental equation results in,

$$\xi \tan \left[(2a+1) \xi \right] = C \quad (3.2-38)$$

If C is small, the solution is approximately,

$$\xi \approx \sqrt{\frac{C}{2a+1}} \quad (3.2-39)$$

but if C is large,

$$\xi \approx \frac{\frac{\pi}{2}}{(2a+1) + \frac{1}{C}} \quad (3.2-40)$$

Interestingly, for large a , Eq. (3.2-40) is consistent with Eq. (3.2-35) if b is at the either end of the array so, for large C and large a , these formulas agree.

Returning now to Eq. (3.2-21), the poles are easily found by iterative bisection because they are all on the negative real axis. The residues are easily computed once the poles are known and the residue series gives the inverse Laplace transform. As an example, this inverse transform is plotted in Fig. 3-5 for the case where $a = 10$, $b = 5$, $C_0 = 1$, and $C = 10$. The time constant of this array is 96.12 inverse locking ranges, whereas the approximate formula Eq. (3.2-36) gives 103.75 inverse locking ranges. Note that for the injected oscillator $x = 5$, the response is much faster than that of the entire array. This is because for this oscillator, the residues of the poles close to the origin of the s plane are small and the more distant poles hold sway.

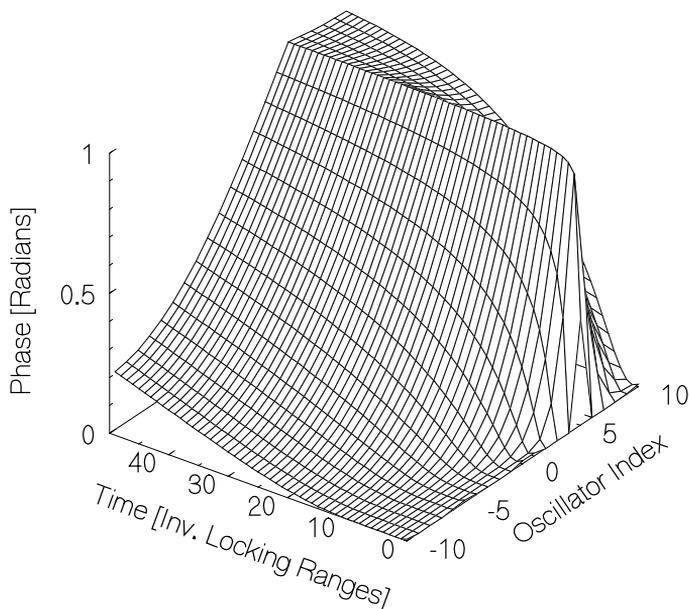


Fig. 3-5. Oscillator phases for oscillator 5 externally injected.

As discussed in connection with the detuned linear array, the above analysis can also be performed by expanding the solution in eigenfunctions of the differential operator. The relevant operator in this case is,

$$\frac{\partial^2}{\partial x^2} - C\delta(x-b) \tag{3.2-41}$$

The presence of the delta function produces a slope discontinuity in the eigenfunctions which must satisfy,

$$\frac{\partial^2 w_n}{\partial x^2} - C\delta(x-b)w_n = \lambda_n w_n \tag{3.2-42}$$

and the boundary conditions,

$$\left. \frac{dw_n}{dx} \right|_{x=a} = 0 \tag{3.2-43}$$

$$\left. \frac{dw_n}{dx} \right|_{x=-a} = 0 \tag{3.2-44}$$

The solution is postulated in the form,

$$w_n = C_b e^{-\sqrt{\lambda_n}|x-b|} + C_R e^{-x\sqrt{\lambda_n}} + C_L e^{x\sqrt{\lambda_n}} \tag{3.2-45}$$

Now we note something interesting about Eq. (3.2-42); that is, it is essentially Eq. (3.2-6) with C_0 set to zero and s set to λ_n . Therefore, we can obtain the eigenfunctions by means of a limiting process applied to Eq. (3.2-21) instead of solving for the three constants using Eqs. (3.2-42), (3.2-43), and (3.2-44). Suppose we set,

$$C_0 = \alpha \tag{3.2-46}$$

and,

$$s = \lambda_n + \alpha \tag{3.2-47}$$

in Eq. (3.2-21) and take the limit as α approaches zero where λ_n is the n^{th} value of s for which $D(s)$ equals zero. In this limit both the numerator and

denominator of Eq. (3.2-21) approach zero, but the ratio is finite and approaches w_n . That is,

$$w_n(x) = \left. \frac{\frac{\partial}{\partial \alpha} \left\{ \alpha C \cosh \left[(2a+1-|x-b|)\sqrt{\lambda_n + \alpha} \right] + \alpha C \cosh \left[(x+b)\sqrt{\lambda_n + \alpha} \right] \right\}}{2 \frac{\partial}{\partial \alpha} \left[(\lambda_n + \alpha) D(\lambda_n + \alpha) \right]} \right|_{\alpha=0} \quad (3.2-48)$$

But, except for a factor of C_0 , this is nothing but the residue of Eq. (3.2-21) at the n^{th} pole. Not only have we found the eigenfunctions, but they are already multiplied by the coefficients needed to form the solution by summation except for an overall multiplicative constant of C_0 . In effect, in Eq. (3.2-48) we are computing,

$$w_n(x) = \frac{C f_n(b)}{\lambda_n \langle f_n, f_n \rangle} f_n(x) \quad (3.2-49)$$

where the bracketed expression in the denominator is the normalization integral; that is, the integral of the square of the arbitrarily normalized eigenfunction, f_n , over the array length and,

$$f_n(x) = C \cosh \left[(2a+1-|x-b|)\sqrt{\lambda_n} \right] + C \cosh \left[(x+b)\sqrt{\lambda_n} \right] \quad (3.2-50)$$

The desired solution is therefore,

$$\tilde{\varphi}(x,s) = C_0 \sum_n w_n(x) = C_0 C \sum_n \frac{f_n(b) f_n(x)}{\lambda_n \langle f_n, f_n \rangle} \quad (3.2-51)$$

the well-known form of the solution as a sum of eigenfunctions.

Thus, we see that the inverse Laplace transform of the eigenfunction sum representing the solution, $\varphi(x, \tau)$, is just the sum of the residues of Eq. (3.2-21) multiplied by the Laplace transform kernel, $e^{s\tau}$. This same property was evident in the treatment of the linear array with one oscillator detuned. It is the reason why Eqs. (3.1-28) and (3.1-51) are identical. Thus, in the present case, we can rest assured that, had we pursued the eigenfunction expansion approach to completion, the result would have been exactly that plotted in Fig. 3-5. The two approaches, the residue series based on the

eigenfunction sum and the residue series based on the particular integral and complementary function are not just equivalent, they are in fact identical.

3.3 Beam-steering via End Detuning

The beam-steering concept suggested by Liao et al. [28] involves antisymmetric detuning of the end oscillators of the linear array. The phase dynamics produced in this situation can be analyzed by means of the continuum model presented in Section 3.1. Beginning with Eq. (3.1-51), we may superpose two such solutions, one with b equal to minus a and the other with b equal to plus a and with C 's of opposite sign. Let, $C = \Delta\omega_T / \Delta\omega_{lock}$ and,

$$\omega_{tune}(x) = \omega_{ref} + \Delta\omega_T \delta(x-a) - \Delta\omega_T \delta(x+a) \quad (3.3-1)$$

Then we obtain,

$$\varphi(x, \tau) = \frac{\Delta\omega_T}{\Delta\omega_{lock}} \sum_{n=0}^{\infty} \frac{2 \sin \left[b \left(\frac{(2n+1)\pi}{2a+1} \right) \right] \sin \left[x \left(\frac{(2n+1)\pi}{2a+1} \right) \right]}{(2a+1) \left(\frac{(2n+1)\pi}{2a+1} \right)^2} \left(1 - e^{-\left(\frac{(2n+1)\pi}{2a+1} \right)^2 \tau} \right) \quad (3.3-2)$$

The steady-state phase distribution is then given by,

$$\varphi(x, \infty) = \frac{\Delta\omega_T}{\Delta\omega_{lock}} \sum_{n=0}^{\infty} \frac{2 \sin \left[b \left(\frac{(2n+1)\pi}{2a+1} \right) \right] \sin \left[x \left(\frac{(2n+1)\pi}{2a+1} \right) \right]}{(2a+1) \left(\frac{(2n+1)\pi}{2a+1} \right)^2} \quad (3.3-3)$$

which can be summed in closed form to yield,

$$\varphi(x, \infty) = \frac{\Delta\omega_T}{\Delta\omega_{lock}} x \quad (3.3-4)$$

a linear phase distribution as indicated in Ref. [28].

The function given by Eq. (3.3-2) is plotted in Fig. 3-6 for end oscillators of a 21-oscillator array step detuned at time zero by one half locking range.

Figure 3-7 shows the corresponding far-zone radiated field if the oscillator outputs are used to excite the elements of a half wavelength spaced array of isotropically radiating elements. It shows that the beam is steered from normal to the array initially, to 9.16 deg from normal corresponding to the steady-state inter-element phase difference of a half radian or 28.65 deg given by Eq. (3.3-4) when $\Delta\omega_T = \frac{1}{2}\Delta\omega_{lock}$. The linearization of the sine functions in the full nonlinear theory introduces some error, but the qualitative behavior is well represented. In fact, the actual steady-state inter-element phase difference is 30 deg resulting in beam-steering to 9.59 deg rather than the 9.16 deg given by the linearized theory.

These plots depict the dynamic behavior for an interval just a little longer than one array time constant.

We have shown that the beam-steering scheme suggested by Liao and York [28] is indeed treatable using the continuum model of coupled oscillators and that the phase transient ensuing from antisymmetric step detuning of the end oscillators produces a smoothly scanning beam in the far zone. The maximum

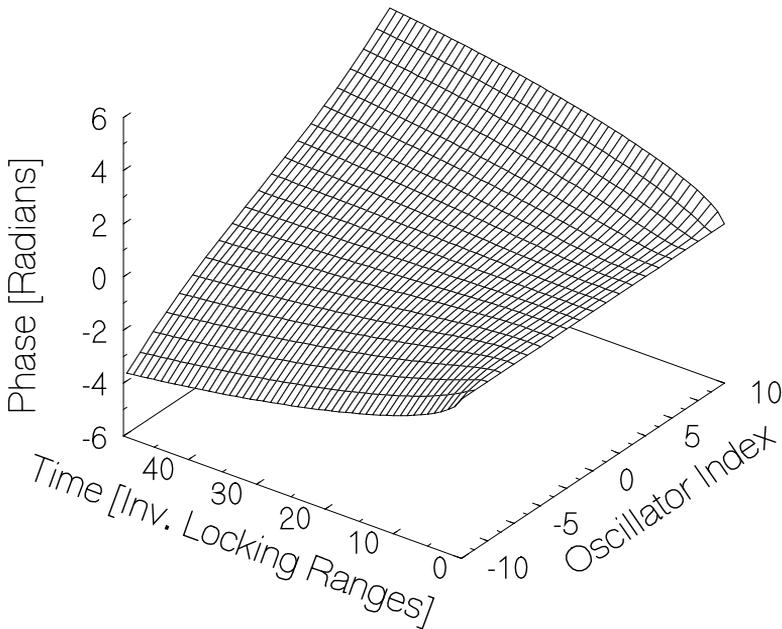


Fig. 3-6. Oscillator phases for a 21-oscillator linear array with end elements antisymmetrically detuned by half the locking range.

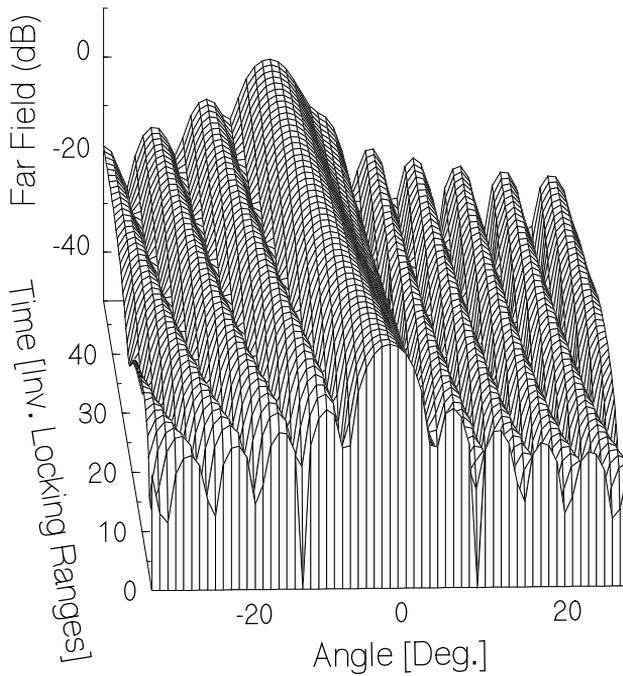


Fig. 3-7. Far-zone radiated field of a 21-element half wavelength spaced phased array excited by the oscillators of Fig. 3-6.

scan angle is limited by the maximum permissible inter-oscillator phase difference. However, this can be mitigated by frequency multiplication of the oscillator outputs, which similarly multiplies the phase excursion [40].

3.4 Beam-steering via End Injection

The beam-steering scheme proposed by Stephan [1] requires that each of the end oscillators be externally injected. The phase distribution across the array is then controlled by adjusting the relative phase of these injection signals by means of a phase shifter which thus controls the beam direction. The dynamic behavior in this situation can be analyzed using the continuum model, but the analysis presented in Section 3.2 for a single injection point cannot be directly applied. If, for example, we represent the solution as a sum of eigenfunctions, the eigenfunctions for two injection points differ from those for one. Similarly, if we approach the analysis using a particular integral and complementary function, both of these will differ from those for one injection point. Thus, it will be necessary to reformulate the problem for two injection points from the beginning.

To be definite, we assume that the oscillators of the array are all initially tuned to the reference frequency and are thus in-phase with each other and that two arbitrary oscillators in the array at $x = b_1$ and $x = b_2$ are injection locked to external signals which are initially in-phase with the oscillators of the array and that at time zero the phase of each of these signals is stepped to a finite constant value. The strengths of the two injection signals are denoted by B_1 and B_2 , and the amplitude of the corresponding temporal step functions are denoted by p_1 and p_2 , respectively. Then, Eq. (3.2-3) becomes,

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x^2} - [B_1 \delta(x - b_1) + B_2 \delta(x - b_2)] \varphi - \frac{d\varphi}{d\tau} \\ = -B_1 \delta(x - b_1) p_1 u(\tau) - B_2 \delta(x - b_2) p_2 u(\tau) \end{aligned} \quad (3.4-1)$$

Laplace transformation results in,

$$\begin{aligned} \frac{\partial^2 \tilde{\varphi}}{\partial x^2} - [B_1 \delta(x - b_1) + B_2 \delta(x - b_2)] \varphi - s \tilde{\varphi} \\ = -B_1 \delta(x - b_1) \frac{p_1}{s} - B_2 \delta(x - b_2) \frac{p_2}{s} \end{aligned} \quad (3.4-2)$$

Now, as shown previously, we may solve this equation either by means of an eigenfunction expansion or by means of superposition of a particular integral and a complementary function. In the former approach, the complexity arises in the normalization of the eigenfunctions, which involves integration of the square of the eigenfunctions of the array. In the latter, this is automatically taken care of by the residues. Thus, we elect to proceed with the latter approach as was done in [39].

The solution of (3.4-2) is postulated in the form,

$$\tilde{\varphi} = C_1 e^{-\sqrt{s}|x-b_1|} + C_2 e^{-\sqrt{s}|x-b_2|} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} \quad (3.4-3)$$

The four unknown constants are determined by the boundary conditions at the array ends, Eqs. (3.2-19) and (3.2-20), and the conditions on the derivatives at the injection points, Eq. (3.2-18). These four constraints yield four equations for the four unknowns in Eq. (3.4-3). The solution is,

$$\begin{aligned}
\tilde{\varphi}(x, s) = & \frac{1}{sD_2(s)} \\
& \times \left\{ B_2 p_2 \cosh \left[\frac{\sqrt{s}}{2} ((2a+1) + (b_2+x) - |b_2-x|) \right] \right. \\
& \quad \times \cosh \left[\frac{\sqrt{s}}{2} ((2a+1) - (b_2+x) - |b_2-x|) \right] \\
& + B_1 p_1 \cosh \left[\frac{\sqrt{s}}{2} ((2a+1) + (b_1+x) - |b_1-x|) \right] \\
& \quad \times \cosh \left[\frac{\sqrt{s}}{2} ((2a+1) - (b_1+x) - |b_1-x|) \right] \\
& + \frac{B_1 B_2 p_2}{\sqrt{s}} \cosh \left[\frac{\sqrt{s}}{2} (2a+1+2b_1) \right] \\
& \quad \times \cosh \left[\frac{\sqrt{s}}{2} (2a+1 - (b_2+b_1) - |b_2-x| - |b_1-x|) \right] \\
& \quad \times \sinh \left[\frac{\sqrt{s}}{2} ((b_2-b_1) - |b_2-x| + |b_1-x|) \right] \\
& + \frac{B_1 B_2 p_1}{\sqrt{s}} \cosh \left[\frac{\sqrt{s}}{2} (2a+1-2b_2) \right] \\
& \quad \times \cosh \left[\frac{\sqrt{s}}{2} (2a+1 + (b_2+b_1) - |b_2-x| - |b_1-x|) \right] \\
& \quad \times \sinh \left[\frac{\sqrt{s}}{2} ((b_2-b_1) + |b_2-x| - |b_1-x|) \right] \left. \right\} \tag{3.4-4}
\end{aligned}$$

where,

$$\begin{aligned}
D_2(s) = & \sqrt{s} \sinh \left[(2a+1)\sqrt{s} \right] \\
& + B_1 \cosh \left[\sqrt{s} \left(a + \frac{1}{2} + b_1 \right) \right] \cosh \left[\sqrt{s} \left(a + \frac{1}{2} - b_1 \right) \right] \\
& + B_2 \cosh \left[\sqrt{s} \left(a + \frac{1}{2} + b_2 \right) \right] \cosh \left[\sqrt{s} \left(a + \frac{1}{2} - b_2 \right) \right] \\
& + \frac{B_1 B_2}{\sqrt{s}} \sinh \left[\sqrt{s} (b_2 - b_1) \right] \\
& \times \cosh \left[\sqrt{s} \left(a + \frac{1}{2} + b_1 \right) \right] \cosh \left[\sqrt{s} \left(a + \frac{1}{2} - b_2 \right) \right]
\end{aligned} \tag{3.4-5}$$

Note that, if either of the B 's is zero, we recover Eqs. (3.2-21) and (3.2-22) for a single injection point. The form of the solution presented in Ref. [39] is slightly different but fully equivalent except for a typographical error in the $\sinh \left[\sqrt{s} (2b_1 - |b_2 - x|) \right]$ term, which should have been $\sinh \left[\sqrt{s} (2b_1 + |b_2 - x|) \right]$. The pole locations on the negative real axis of the s plane are easily found by iterative bisection, and the inverse Laplace transform is then obtainable as a residue series.

As a first example, we compute the solutions when unit strength injection signals are applied to the end oscillators of a 21-oscillator linear array, and at time zero their phase is step shifted antisymmetrically by one radian producing a phase difference of two radians. The dynamic behavior of the resulting phase distribution is shown in Fig. 3-8.

An analytic expression for the steady-state solution for the phase can be obtained by application of the final value theorem to the transform (3.4-4) and (3.4-5). The result is,

$$\begin{aligned}
\varphi(x, \infty) = & \frac{1}{2B_1 + 2B_2 + 2B_1 B_2 (b_2 - b_1)} \left\{ 2B_1 p_1 + 2B_2 p_2 \right. \\
& \left. + B_1 B_2 \left[(p_2 + p_1)(b_2 - b_1) - (p_2 - p_1)(|b_2 - x| - |b_1 - x|) \right] \right\}
\end{aligned} \tag{3.4-6}$$

For the case shown in Fig. 3-8, this expression reduces to,

$$\varphi(x, \infty) = \frac{x}{11} \tag{3.4-7}$$

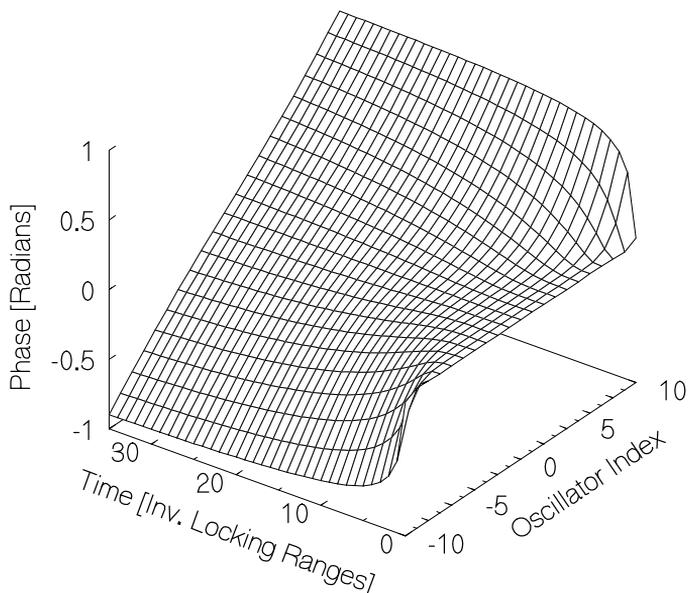


Fig. 3-8. Phase dynamics for injected end elements.

Notice that the steady-state phases of the injected oscillators at $x = 10$ and at $x = -10$ are not equal to the phases of the corresponding injection signals, plus and minus one radian. This is because the end oscillators are also injected by virtue of their coupling to their nearest neighbor in the array, and the phase of that neighbor differs from the phase of the external injection signal. Thus, the total injection of the end oscillator is not in phase with the external injection signal. However, as the strength of the injection signals is increased (large values of the B 's are used), the steady-state phase of the end oscillators will approach the phase of the corresponding injection signals because the signal from the corresponding neighboring oscillators becomes negligible.

We again remark, as in Section 2.5, that the injection signals may be derived from the end oscillators of the array and used to inject the next to end oscillators to achieve beam-steering. The continuum model has been used to study this approach also [41].

Recall that the phase of the injection signals can differ from the initial phase of the injected oscillators by no more than $\pi/2$ radians for a maximum total phase difference of π radians across the array. Thus, for strong injection, the beam-steering angle is limited to a maximum of

$$\theta_{\max} = \sin^{-1} \left(\frac{\pi}{(2a+1)S} \right) \quad (3.4-8)$$

where S is the electrical radiating element spacing in radians. In our present example, if the element spacing is a half wavelength so $S = \pi$, then the maximum steering angle is 2.73 deg, a disappointingly small angle. Fortunately, this problem is easily eliminated by gradually increasing the injection phase instead of stepping it. [1] That way, the phase difference between the injected oscillator and the injection signal can be maintained less than $\pi/2$ radians while the phase difference between the two injection signals is increased to a large value. The new limit on steering angle is now imposed by the requirement that the inter-oscillator phase difference be less than $\pi/2$ radians to maintain overall lock, a limitation also present in the detuning case. In the present example, this limits the steering angle to 30 degrees, a certainly more acceptable limit.

As an example of this enhanced beam-steering scheme, we compute the response of the array of the previous example, but this time we gradually increase the injection-signal phase difference by convolving the step function with a temporal Gaussian. By virtue of the linearity of the p dependence of the equation, we may obtain the corresponding phase response by convolving the step response with the same Gaussian. Since the solution is a residue series, each term has simple exponential time dependence so the convolution can be carried out analytically term by term as described in detail in Ref. [39].

Let the Gaussian be,

$$g(\tau) = e^{-(\tau-6)^2/100} \quad (3.4-9)$$

Then, setting p_2 equal to 2π radians and p_1 equal to -2π radians for a total phase difference of 4π radians, the expected steady-state beam-steering angle of a half wavelength spaced array will be 10.48 deg. The steady-state inter-oscillator phase difference is 0.628 radians, for which the sine functions are approximated by their argument with about 7-percent accuracy. However, there are times during the transient at which this difference becomes as large as 0.878 radians near the array ends. At these times, the sine functions are approximated with only 14-percent accuracy. Thus, the actual inter-oscillator phase difference will be somewhat larger. The phase behavior for these parameters and unit amplitude injection as predicted by the continuum model is shown in Fig. 3-9, and the corresponding far zone beam is shown in Fig. 3-10.

We have shown the utility of the continuum model in analyzing the transient behavior of linear arrays of mutually injection locked oscillators with external

injection. Beam-steering of linear phased arrays of radiating elements can be achieved by externally injecting the end oscillators of the array and varying the relative phase if the injection signals as suggested by Stephan [1]. In order to achieve significant beam-steering angles via this approach, it is necessary to apply the phase shift to the injection signals gradually so as to avoid excessive inter-oscillator phase differences resulting in loss of lock. Here, as in the detuning approach, the steering angle range may be extended via frequency multiplication.

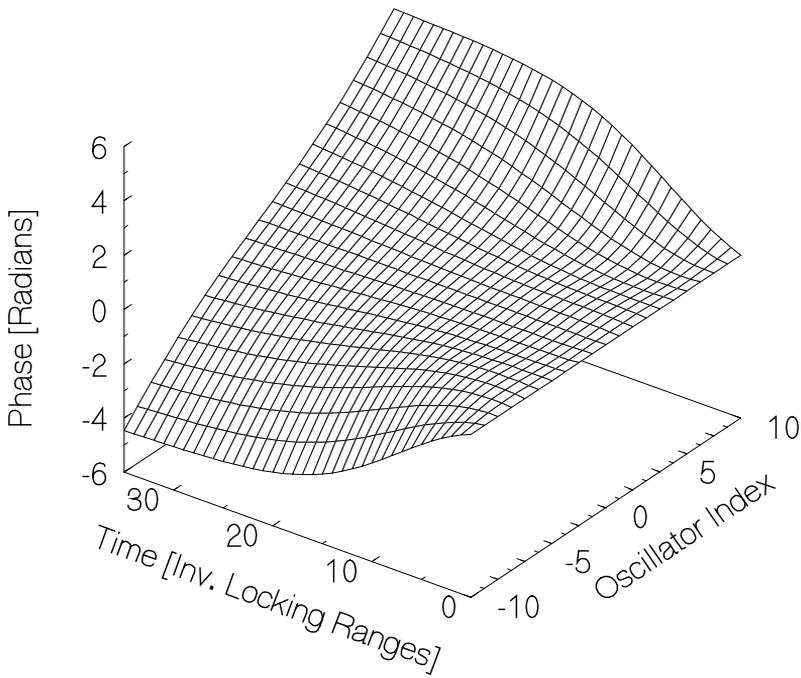


Fig. 3-9. Phase dynamics for gradually changing injection phase.

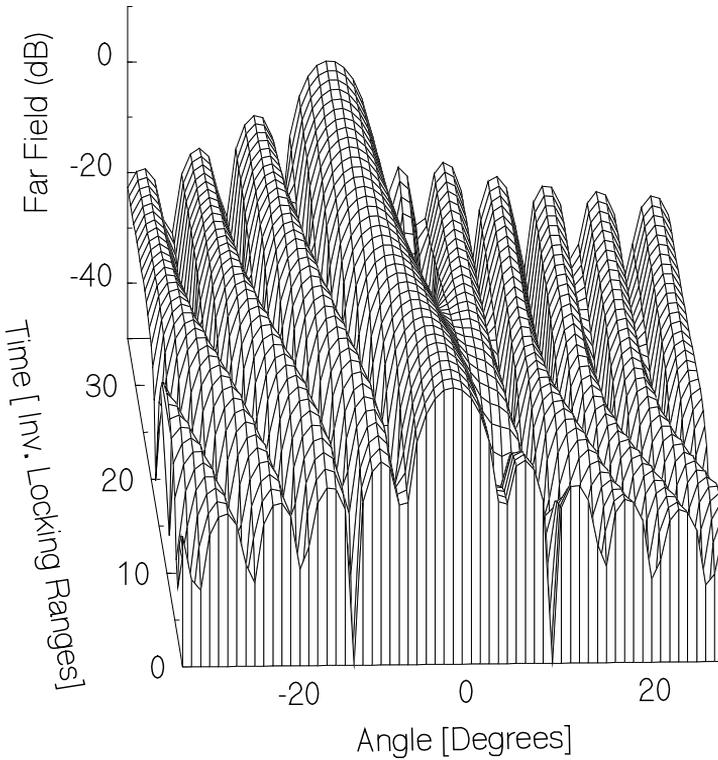


Fig. 3-10. Far-zone radiated field of a 21-element half-wavelength-spaced phased array excited by the oscillators of Fig. 3-9.

3.5 Conclusion

In this chapter, the continuum model was shown to provide considerable physical insight into the general behavior of one-dimensional coupled oscillator arrays. It highlights the fact that the phase behavior is governed by the diffusion equation, and as a consequence, the transient response time is proportional to the square of the array length. In the next chapter we extend the continuum model to planar arrays. This broadens the nearest neighbor coupling concept to a wider range of topologies. That is, in the planar case we can envision not only the Cartesian scheme discussed in Chapter 2, in which each oscillator is coupled to its four nearest neighbours, but also hexagonal and triangular schemes in which each oscillator is coupled to three or six nearest neighbours, respectively. By means of the continuum model, we will see that these coupling topologies produce similar phase behavior but result in differing response times for the arrays.